C_p is the specific heat at constant pressure for the gas
C_v is the specific heat at constant volume for the gas

5 – 2 – Changes in Fundamental Properties of Perfect Gas Undergoing a Reversible Polytropic or Isentropic Flow Process

The relation between the potential head z, the pressure P, the density ρ and the velocity C of a flowing non-viscous fluid was given by equation (3.44) as follows:

$$g\,dz + \frac{dP}{\rho} + C\,dC = 0$$

If there is no change in the potential head z, this equation reduces to:

$$\frac{dP}{\rho} + C\,dC = 0 \tag{5.12}$$

From equation (5.11):

$$dP = R(\rho d\,T + T\,d\,\rho) \tag{5.13}$$

Equations (5.12) and (5.13) combined, give;

$$\frac{R\rho dT}{\rho} + \frac{RTd\rho}{\rho} + C\,dC = 0 \tag{5.14}$$

Substituting from equation (5.4) into (5.14) and integrating we get:

$$RT + \frac{RK4}{n-1}\,\rho^{n-1} + \frac{C^2}{2} = \text{constant} \tag{5.15}$$

which may be reduced to the form;

$$\left(\frac{nR}{n-1}\right) T + \frac{C^2}{2} = \text{constant} \tag{5.16}$$

Equation (5.16) can be written in the form

$$\left(\frac{nR}{n-1}\right) T_1 + \frac{C_1^2}{2} = \left(\frac{nR}{n-1}\right) T_2 + \frac{C_2^2}{2} \tag{5.17}$$

which may be reduced to the form;

$$\frac{T_2}{T_1} = 1 - \frac{n-1}{nRT_1} \left(\frac{C_2^{\,2} - C_1^{\,2}}{2} \right) \qquad (5.18)$$

Equation (5.4) also gives;

$$\frac{T_2}{T_1} = \left(\frac{\rho_1}{\rho_2} \right)^{1-n} = \left(\frac{\rho_2}{\rho_1} \right)^{n-1} \qquad (5.19)$$

From which into (5.18) we get;

$$\frac{\rho_2}{\rho_1} = \left[1 - \frac{n-1}{nRT_1} \left(\frac{C_2^{\,2} - C_1^{\,2}}{2} \right) \right]^{\frac{1}{n-1}} \qquad (5.20)$$

and equation (5.5) gives;

$$\frac{T_2}{T_1} = \left(\frac{P_1}{P_2} \right)^{\frac{1-n}{n}} = \left(\frac{P_2}{P_1} \right)^{\frac{n-1}{n}} \qquad (5.21)$$

From which into (5.18) we get;

$$\left(\frac{P_2}{P_1} \right) = \left[1 - \frac{n-1}{nRT_1} \left(\frac{C_2^{\,2} - C_1^{\,2}}{2} \right) \right]^{\frac{n}{n-1}} \qquad (5.22)$$

If the process undergone by the gas is isentropic, equations (5.17), (5.18), (5.20) and (5.22) to be the following forms respectively;

$$\left(\frac{\gamma R}{\gamma - 1} \right) T_1 + \frac{C_1^{\,2}}{2} = \left(\frac{\gamma R}{\gamma - 1} \right) T_2 + \frac{C_2^{\,2}}{2} \qquad (5.23)$$

$$\frac{T_2}{T_1} = \left[1 - \frac{\gamma - 1}{\gamma R T_1} \left(\frac{C_2^{\,2} - C_1^{\,2}}{2} \right) \right] \qquad (5.24)$$

$$\frac{\rho_2}{\rho_1} = \left[1 - \frac{\gamma - 1}{\gamma R T_1} \left(\frac{C_2^{\,2} - C_1^{\,2}}{2} \right) \right]^{\frac{1}{\gamma - 1}} \qquad (5.25)$$

$$\frac{P_2}{P_1} = \left[1 - \frac{\gamma - 1}{\gamma R T_1} \left(\frac{C_2^{\,2} - C_1^{\,2}}{2} \right) \right]^{\frac{\gamma}{\gamma - 1}} \qquad (5.26)$$

EXAMPLE 5.1

Air flowing through a heat exchanger has its pressure reduced from 200 kN/m^2 to 195 kN/m^2. Its velocity and temperature at the inlet to the heat exchanger are 100 m/s and 35°C respectively. Assuming air to be a perfect gas with specific gas constant R=0.287 kJ/kg.K and the polytropic expansion index of the process is 1.6, calculate the velocity and temperature of the air at the exit of the heat exchanger.

Solution

$$\frac{T_2}{T_1} = \left(\frac{P_2}{P_1}\right)^{\frac{n-1}{n}} = 1 - \frac{-1}{nRT_1}\left(\frac{C_2^2 - C_1^2}{2}\right)$$

$$\left(\frac{195}{200}\right)^{\frac{1.6-1}{1.6}} = 1 - \frac{(1.6-1)}{1.6\times287\times308\times2}\left(C_2^2 - C_1^2\right)$$

$$(0.975)^{0.375} = 1 - \frac{0.6}{282867}\left(C_2^2 - C_1^2\right)$$

$$0.99055 - 1 = \frac{0.6}{282867}\left(C_2^2 - C_1^2\right)$$

$$-\left(C_2^2 - C_1^2\right) = \frac{-0.00945\times282867}{0.6} = -4454.8$$

$$C_2^2 = C_1^2 + 4454.8 = 10000 + 4454.8$$
$$= 14454.8$$

$$C_2 = 120.2 \text{ m/s}$$

$$T_2 = 0.99055\ T_1 = 0.99055\times308$$
$$= 305.8 \text{ K}$$

5-3- The Speed of Propagation of Weak Pressure Wave in Gas (Acoustic Speed)

An object suddenly introduced into a gas stream would cause various changes in pressure around its profile. Such pressure changes react on the fluid continuum around the object.

A pressure disturbance of this kind transmitting itself further and further away from the origin affects progressively greater amounts of the fluid and in consequence the pressure wave propagating away from the source becomes progressively weaker. By the time the pressure disturbance makes itself felt at a great distance from its origin, the pressure change has reduced to a very small value δP. The speed at which the weak pressure wave propagates relative to the fluid is denoted by "a". Since sound is believed to propagate in gases in this manner,this speed is known as "the speed of sound" or "the acoustic speed".

Now let us consider the occurance of such pressure in a constant area stream tube Fig. (5.1)

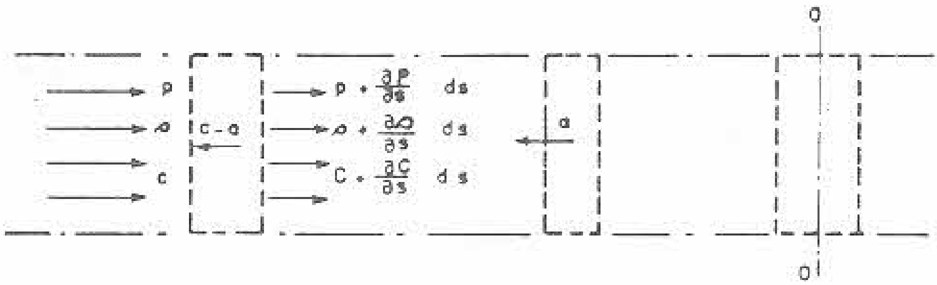


Fig. 5.1 Pressure wave in a constant area stream tube

If the origin of a pressure disturbance is at 0 – 0, the pressure wave would travel in all direction at a speed "a". Assuming that the flow is travelling with a velocity "C" in the direction shown in Fig. (5.1), then the pressure wave would be travelling upstream at a velocity "C-a".

Now considering the pressure wave itself to be a control volume then at its front the fluid has not yet felt its effect and would be crossing the front boundary of the wave at a velocity C, a pressure P and a density ρ. At the rear boundary of the wave the fluid after being affected by the pressure wave would assume new conditions different from those at the inlet to the wave, that is a velocity; $C + \frac{\partial C}{\partial s} ds$, a pressure $p + \frac{\partial p}{\partial s} ds$ and a density; $\rho + \frac{\partial \rho}{\partial s} ds$. The propagation of the press ure wave assumed to be steady renders the time factor to be of no effect.

Applying continuity and momentum rules to the wave as a control volume, bearing in mind that the wave neither stores nor gives away mass, we obtain;
i) by continuity

$$[C-(C-a)]\rho .A=[(C + \frac{\partial c}{\partial s} ds) - (C-a)](\rho + \frac{\partial \rho}{\partial s} ds)A \tag{5.27}$$

$$(C-C+a)\rho = (C + \frac{\partial c}{\partial s} ds - C+a)(\rho + \frac{\partial \rho}{\partial s} ds) \tag{5.28}$$

or

$$\rho a = (a + \frac{\partial c}{\partial s} ds)(\rho + \frac{\partial \rho}{\partial s} ds) \tag{5.29}$$

i.e.

$$\rho a = \rho a + \rho \frac{\partial c}{\partial s} ds + a \frac{\partial \rho}{\partial s} ds + \frac{\partial c}{\partial s} ds \times \frac{\partial \rho}{\partial s} ds \tag{5.30}$$

neglecting second order terms this yields;

$$\rho .\partial c = - a \partial \rho \tag{5.31}$$

ii) and from momentum considerations we have,

| Time rat of change of momentum of control volume in direction | $_{x, y \text{ or } z}$ = | Time rate of momentum influx into the control volume in direction | $_{x, y \text{ or } z}$ |

− | Time rate of momentum efflux out of control volume in direction | $_{x, y \text{ or } z}$ + | Sum of external forces (including forces induced by control volume like static pressures...etc) on control volume in direction | $_{x, y \text{ or } z}$

so that;

$$\frac{d}{dt}(mC_{x \text{ y or z}}) = \int d\dot{m}_i \, C_{xi \text{ yi or zi}} - \int d\dot{m}_o \, C_{xo \text{ yo or zo}} + \Sigma \bar{F}_{x \text{ y or z}} \tag{5.32}$$

Considering one direction for the above analysis the above equation (5.32) becomes;

$$m \frac{dc_x}{dt} + C_x \frac{dm}{dt} = \int d\dot{m}_i \; C_{xi} - \int d\dot{m}_o \; C_{xo} + \Sigma F_x \qquad (5.33)$$

For the pressure wave m is constant, and $C_x = a$ also constant. The speed of sound in the model given would not be a function of time. In other words if the source keeps emitting the pressure waves, the sound would travel at a particular pattern in the medium. The speed of sound may differ from one place to the other in the medium but it would not be a function of time.

Therefore

$$\frac{dC_x}{dt} = 0 \quad , \qquad \frac{dm}{dt} = 0$$

$$\int d\dot{m}_i \quad C_{xi} = \rho A[C-(C-a)]C = \rho a \; AC \qquad (5.34)$$

$$\int d\dot{m}_o \quad C_{xo} = [C + \frac{\partial c}{\partial s} ds - (C-a)](\rho + \frac{\partial \rho}{\partial s} ds)A(c + \frac{\partial c}{\partial s} ds)$$

$$= A(a + \frac{\partial c}{\partial s} ds)(\rho + \frac{\partial \rho}{\partial s} ds)(c + \frac{\partial c}{\partial s} ds)$$

$$= \rho a \; Ac + acA \frac{\partial \rho}{\partial s} ds + \rho cA \frac{\partial c}{\partial s} ds + a\rho A \frac{\partial c}{\partial s} ds \qquad (5.35)$$

$$\Sigma F_x = \; A - (p + \frac{\partial p}{\partial s} ds) \; A = - A \frac{\partial p}{\partial s} ds \qquad (5.36)$$

substituting these quantities in equation (5.33) we get

$$0 = \rho aAc - \rho aAc - acA \frac{\partial \rho}{\partial s} ds - \rho cA \frac{\partial c}{\partial s} ds$$

$$- a\rho A \frac{\partial c}{\partial s} ds - A \frac{\partial p}{\partial s} ds \qquad (5.37)$$

so that;

$$ac \; \partial\rho + \rho c \; \partial c + a\rho \; \partial c + \partial p = 0 \qquad (5.38)$$

Substituting from equation (5.31) into equation (5.38) we get;

$$ac\ \partial\rho - ac\ \partial\rho - a^2\partial\rho + \partial p = 0 \tag{5.39}$$

i.e. $$\partial p = a^2\ \partial\rho \tag{5.40}$$

or $$a^2 = \frac{\partial p}{\partial\rho} \tag{5.41}$$

Equation (5.41) while giving a value for the speed of sound in the fluid in terms of its pressure and density, indicates that the speed of sound in a fluid is not a function of the fluids own velocity; never the less it may be indirectly so particularly when the density of the fluid changes according to changes in the velocity of the fluid (see equation 5.25)

It is essential to note that no restrictions have been made in the above analysis which means that the conclusion made in equation (5.41) so far applies to any fluid.

However, the speed of sound is relatively high, and the magnitude of time in which a fluid mass is under the effect of the pressure wave is extremely short so that any exchange of heat between their mass and its surroundings during this infinitesimal time would be inconceivable. The process is therefore considered to be *adiabatic.* The pressure wave taking the form of an elastic control volume makes it also fessible to assume that the process is *reversible.*

Thus the propagation of sound in gases may be assumed isentropic, for which equation (5.7) gives;

$$P = K_7\ \rho^{\gamma} \tag{5.42}$$

that is; P is a function of ρ only and equation (5.41) may now be written in the form

$$\frac{dp}{d\rho} = a^2 \tag{5.43}$$

and from equation (5.41) we have;

$$\frac{dp}{d\rho} = \gamma K_7\ \rho^{\gamma-1} \tag{5.44}$$

and since

$$K_7 = \frac{P}{\rho\gamma}$$

therefore

$$\frac{dp}{d\rho} = \frac{\gamma P\rho^{\gamma-1}}{\rho\gamma} = \frac{\gamma P}{\rho} \tag{5.45}$$

Now equations (5.43) and (5.45) yield.

$$a^2 = \frac{\gamma P}{\rho} \tag{5.46}$$

and from equation (5.11) we have

$$\frac{P}{\rho} = RT$$

from which into equation (5.46) we obtain;

$$a^2 = \gamma RT \tag{5.47}$$

or

$$a = \sqrt{\gamma RT} \tag{5.48}$$

values for R and γ for common gases are given in Table 5.1

Table 5.1 Gas constant, R, and specific heat ratio, γ, at 298.15 K for common gases*

Gas	chemical symbol	R J/kg. K	γ
Air		287.06	1.4
Argon	A	208.15	1.654
Carbon monoxide	CO	296.83	1.398
Carbon dioxide	CO_2	188.92	1.288
Helium	He	2078.2	1.659
Hydrogen	H_2	4124.2	1.405
Methane	CH_4	518.25	1.304
Nitrogen	N_2	296.8	1.4
Oxygen	O_2	259.82	1.395
Water	H_2O	461.5	1.329

(*) Taken from: Zucrow, M.J. and Hoffman, J.D., Gas Dynamics, John Wiley & Sons Inc., 1976.

EXAMPLE 5.2

Calculate the speed of sound in air at $-50°C$, $0°C$, $50°C$ and $100°C$

Data of the problem
* Fluid = air

Requirements

* Speed of sound at $-50°C, 0°C, 50°C$ and $100°C$

Solution

For air $R = 287 \quad J/kg.K$

$\gamma = 1.4 \quad \text{for } T < 600 \text{ K}$

Substituting the above values into equation (5.48) we get

$$a = \sqrt{\gamma RT} = \sqrt{1.4 \times 287 \times T} = 20.045\sqrt{T}$$

Substituting values of temperature in Kelvin in this equation we get the requirements as tabulated below;

t.°c	T.K	a m/s
-50	223	299.3
0	273	331.2
50	323	360.3
100	373	387.1

5 – 4 – Evaluation of Fundamental Properties of a Perfect Gas at Stagnation

Quite often the measurement of the velocity of a compressible fluid involves bringing a streamline of this fluid to stagnation and measuring one or more if its fundamental properties. The process of stagnation takes very short time so that it may be considered isentropic.

The equations (5.24), (5.25) and (5.26), if the fluid at point 2 is brought to stagnation i.e. $C_2 = 0$ and the quantity γRT_1 is substituted by a_1^2 these equations take the form;

$$\frac{T_o}{T_1} = \left[1 - \frac{\gamma - 1}{2} \frac{C_1^2}{a_1^2} \right] \qquad (5.49)$$

$$\frac{\rho_o}{\rho_1} = \left[1 + \frac{\gamma - 1}{2} \frac{C_1^2}{a_1^2} \right]^{1/\gamma - 1} \qquad (5.50)$$

$$\frac{P_o}{P} = \left[1 + \frac{\gamma - 1}{2} \frac{C_1^2}{a_1^2} \right]^{\gamma/\gamma - 1} \qquad (5.51)$$

where "0" refers to stagnation property.

5 – 5 – Subsonic and Supersonic Flows

From equation (5.23) we have;

$$\frac{\gamma}{\gamma-1} RT_1 + \frac{C_1^2}{2} = \frac{\gamma}{\gamma-1} RT_2 + \frac{C_2^2}{2} \tag{5.52}$$

Using equation (5.11) and substituting

$$\frac{P_1}{\rho_1} = RT_1 \quad \text{and} \quad \frac{P_2}{\rho_2} = RT_2$$

in equation (5.52) we get;

$$\frac{\gamma}{\gamma-1} \frac{P_2}{\rho_2} + \frac{C_2^2}{2} = \frac{\gamma}{\gamma-1} \frac{P_1}{\rho_1} + \frac{C_1^2}{2} \tag{5.53}$$

From which stagnation conditions can be given as;

$$\frac{\gamma}{\gamma-1} \frac{P_o}{\rho_o} = \frac{\gamma}{\gamma-1} \frac{P}{\rho} + \frac{C^2}{2} \tag{5.54}$$

If "a" is the speed of sound at pressure "P" and "a_o" is that at "P_o", then we have;

$$a^2 = \frac{\gamma P}{\rho} \quad \text{and} \quad a_o^2 = \frac{\gamma P_o}{\rho_o} \tag{5.55}$$

from which into equation (5.54) we obtain;

$$\frac{a_o^2}{\gamma-1} = \frac{a^2}{\gamma-1} + \frac{C^2}{2} \tag{5.56}$$

which shows that "a" has the maximum value "a_o" when "C" is equal to zero and that "C" has a maximum value "C_{max}" when "a" is equal to zero in which case C_{max} would be given as;

$$C_{max}^2 = \frac{2}{\gamma-1} a_o^2 \tag{5.57}$$

The critical speed occurs when sound speed a_* and fluid velocity C_* are equal and therefore from equation (5.54) we have;

$$\frac{a_o^2}{\gamma-1} = \frac{a_*^2}{\gamma-1} + \frac{C_*^2}{2} \tag{5.58}$$

that is;

$$a_o^2 = a_*^2 + \frac{\gamma-1}{2} a_*^2$$

$$= a_*^2 \left(1 + \frac{\gamma-1}{2}\right)$$

$$= a_*^2 \left(\frac{\gamma+1}{2}\right) \tag{5.59}$$

or

$$a_* = C_* = a_o \sqrt{\frac{2}{\gamma+1}} \tag{5.60}$$

Now substituting for

$$\frac{a_o^2}{\gamma-1} = \frac{1}{2} C_{max}^2$$

into equation (5.58) we obtain;

$$\frac{1}{2} C_{max}^2 = \frac{a^2}{\gamma-1} + \frac{C^2}{2} \tag{5.61}$$

that is

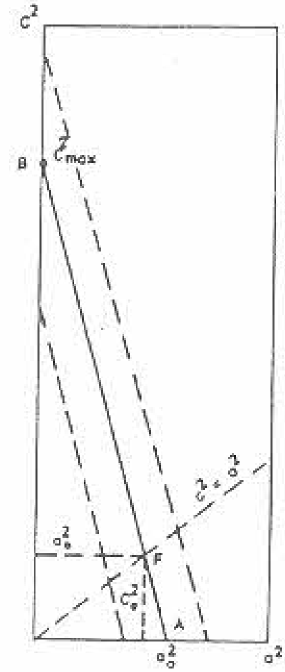


Fig. 5.2 Subsonic and supersonic flows

$$C^2 = -\frac{2}{\gamma-1} a^2 + C_{max}^2 \tag{5.62}$$

Equation (5.62) gives a family of straight lines as shown in Fig. (5.2). Each of these lines indicate that with a specified sum of kinetic and pressure energy, the velocity of the fluid along a particular streamline in a fluid continuum can only reach a maximum given by C_{max} that is, $C \leqslant C_{max}$ and that the speed of sound at any point along this streamline can only reach a maximum given by a_o so that $a \leqslant a_o$.

Now, the straight line $C^2 - a^2 = 0$, gives the locus of the critical speeds; for example it cuts the line AB at the point F (C_*^2, a_*^2) where $C_* = a_*$.

The line OF divides the graph into two regions each corresponding to a physically different regime.

The ratio (C/a) is known as "Mach number" and denoted by M so that;

$$M = C/a \tag{5.63}$$

Then at any point in the region below line OF, we have $C < C_* = a_* < a$ so that $M < 1$, provided that $C < a$. A flow for which $M < 1$ is called subsonic.

At any point in the region above line OF, we have $C > C_* = a_* > a$, so that $M > 1$ and the flow is then said to be supersonic.

From equation (5.56) we now have;

$$\frac{a_o^2}{a^2} = 1 + \frac{\gamma-1}{2}\,\frac{C^2}{a^2} \tag{5.64}$$

or

$$1 + \frac{\gamma-1}{2} M^2 = \frac{a_o^2}{a^2} = \left(\frac{\rho_o}{\rho}\right)^{-1}$$

$$= \left(\frac{P_o}{P}\right)^{\frac{\gamma-1}{\gamma}} = \frac{T_o}{T} \tag{5.65}$$

Now fundamental stagnation properties can be expressed as;

$$T_o = T\left[1 + \frac{\gamma-1}{2} M^2\right] \tag{5.66}$$

$$\rho_o = \rho\left[1 + \frac{\gamma-1}{2} M^2\right]^{\frac{1}{\gamma-1}} \tag{5.67}$$

$$P_o = P\left[1 + \frac{\gamma-1}{2} M\right]^{\frac{\gamma}{\gamma-1}} \tag{5.68}$$

Where M = 1, equations (5.66), (5.67) and (5.68) take the form

$$\frac{T_*}{T_o} = \frac{2}{\gamma+1} \tag{5.69}$$

$$\frac{\rho_*}{\rho_o} = \left(\frac{2}{\gamma+1}\right)^{\frac{1}{\gamma-1}} \tag{5.70}$$

$$\frac{P_*}{P_o} = \left(\frac{2}{\gamma+1}\right)^{\frac{\gamma}{\gamma-1}} \qquad (5.71)$$

The relations between the isentropic flow properties and the Mach number as given by the above equations (5.66 – 5.71) are presented graphically in Fig. (5.3) for air (γ = 1.4)

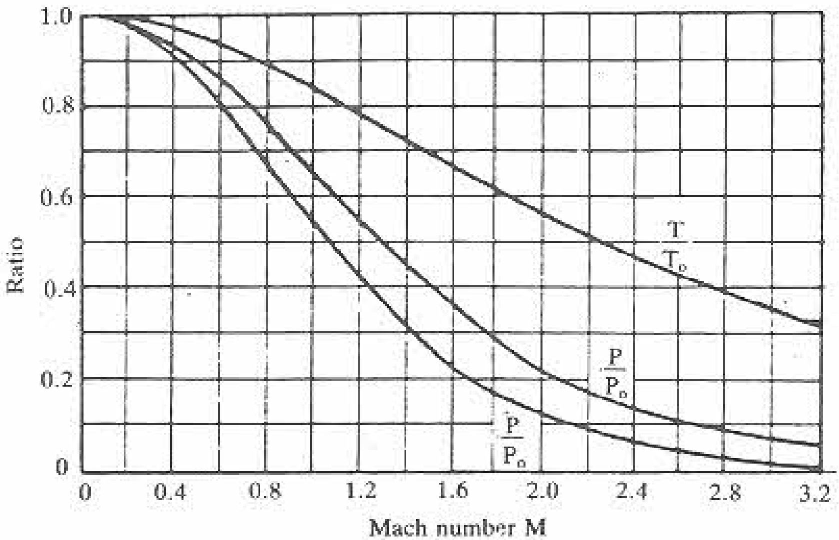


Fig. 5.3 Isentropic flow pressure, density and temperature as a function of Mach number (for air, γ = 1.4)

EXAMPLE 5.3

A gas flow through a passage with a speed of 800 m/s. Its local static temperature is 1527°C, its specific heat ratio $\gamma = 1.25$ and the gas constant $R = 332.8$ J/kg K. Calculate the Mach number.

Data of the problem

* Fluid is a gas
* C = 800m/s
* T = 1257°C = 1800 K
* γ = 1.25
* R = 322.8 J/kg.K

Requirements

* Mach number M.

Solution

Equation (5.48) gives the acoustic speed as

$$a = \sqrt{\gamma R T}$$

Thus,

$$a = \sqrt{1.25 \times 322.8 \times 1800} = 852.2 \quad m/s$$

Equation (5.63) defines Mach number as;

$M = C/a$

Hence

$$M = \frac{800}{852.2} = 0.9387$$

---

5 – 6 – Range of Compressibility

From equation (5.68) we have;

$$\frac{P_o}{P} = [\ 1 + \frac{\gamma - 1}{2} M^2\ ]^{\frac{\gamma}{\gamma - 1}}$$

Expanding this by the binomial theorem;

$$[\text{i.e. } (1+x)^m = i + nx + \frac{n(n-1)}{21} x^2 + \frac{n(n-1)(n-2)}{31} x^3 + \ldots.]$$

we get;

$$\frac{P_o}{P} = 1 + \frac{\gamma}{2} M^2 + \frac{\gamma}{8} M^4 + \frac{\gamma}{4.8} M^6 + \ldots \qquad (5.72)$$

from which the ratio of the third term to the second term in the right hand expansion is

$$(\frac{1}{4} M^2) \text{ i.e. } (\frac{1}{4} \frac{C^2}{a^2})$$

so that even when the speed C is equal to half the speed of sound this ratio is 1/16. Thus it appears that we may to a good approximation neglect the third term unless "C" is a considerable function of the speed of sound. Bernoulli equation for air (for example) would still take the form;

$$\frac{P_o}{\rho_o} = \frac{P}{\rho} + \frac{C^2}{2} \qquad (5.73)$$

which means that air may be treated as incompressible within a very considerable range of speeds. In particular for air speeds of up to 450 km/hr the error in speed measurements made by the use of a pitot tube will be about two percent.

5 – 7 – Effects of Area Variation

From Bernoulli's equation when potential head Z is fixed equation (5.12) give;

$$\frac{dP}{\rho} + C\,dc = 0$$

that is

$$dP = -\rho\,C\,dc = -\rho\,C^2\,\frac{dc}{C} \tag{5.74}$$

Continuity equation (3.16) gives,

$\rho C A$ = constant

In logarithmic form this gives;

$$\log\rho + \log C + \log A = \text{constant} \tag{5.75}$$

In a derivative form this equation gives

$$\frac{d\rho}{\rho} + \frac{dc}{C} + \frac{dA}{A} = 0 \tag{5.76}$$

Equations (5.74) and (5.76) give

$$dP = \rho C^2 \left(\frac{d\rho}{\rho} + \frac{dA}{A} \right) \tag{5.77}$$

Now, if the density change is small (that is, the fluid is incompressible) then $d\rho = 0$ and we get;

$$dP = \rho C^2\,\frac{dA}{A} \tag{5.78}$$

But we have $\rho\,C\,A$ = constant, and since ρ is constant in this case, it follows that, C A = constant, say k,so that;

$$C = \frac{K}{A}$$

that is

$$C^2 = \frac{K^2}{A^2}$$

Thus

$$dP = \rho K^2\,\frac{dA}{A^3} \tag{5.79}$$

Hence, if area A is increasing then dA is positive and dP must be positive, that is pressure increases with area and vice versa.

However as the velocity becomes considerably large, density (particularly for gases) cannot be considered constant and density changes must be taken into consideration.

However, from equation (5.43) we have;

$$d\rho = \frac{dP}{a^2}$$

Putting this into equation (5.77) we get

$$dP = \rho C^2 \left(\frac{dP}{\rho a^2} + \frac{dA}{A} \right) \tag{5.80}$$

that is

$$dP = \frac{C^2 dP}{a^2} + \frac{\rho C^2 dA}{A} = M^2 \; dP + \rho C^2 \frac{dA}{A}$$

or

$$(1-M^2) \; dP = \rho C^2 \frac{dA}{A} \tag{5.81}$$

so that

$$dP = \frac{\rho C^2}{(1-M^2)} \frac{dA}{A} \tag{5.82}$$

The density change may now be given as;

$$d\rho = \frac{dP}{a^2} = \frac{\rho C^2}{a^2} \cdot \frac{1}{(1-M^2)} \frac{dA}{A} \tag{5.83}$$

or

$$\frac{dP}{\rho} = \frac{M^2}{(1-M^2)} \frac{dA}{A} \tag{5.84}$$

Equations (5.76) and (5.84) now give;

$$\frac{dc}{C} = -\frac{d\rho}{\rho} - \frac{dA}{A} = -\frac{M^2}{(1-M^2)} \frac{dA}{A} - \frac{dA}{A} \tag{5.85}$$

Hence,

$$\frac{dc}{C} = -\frac{1}{(1-M^2)} \frac{dA}{A} \tag{5.86}$$

The three equations (5.82), (5.84) and (5.86) express the variation with area of the pressure, density and velocity respectively. These equations are represented schematically in Fig. (5.4).

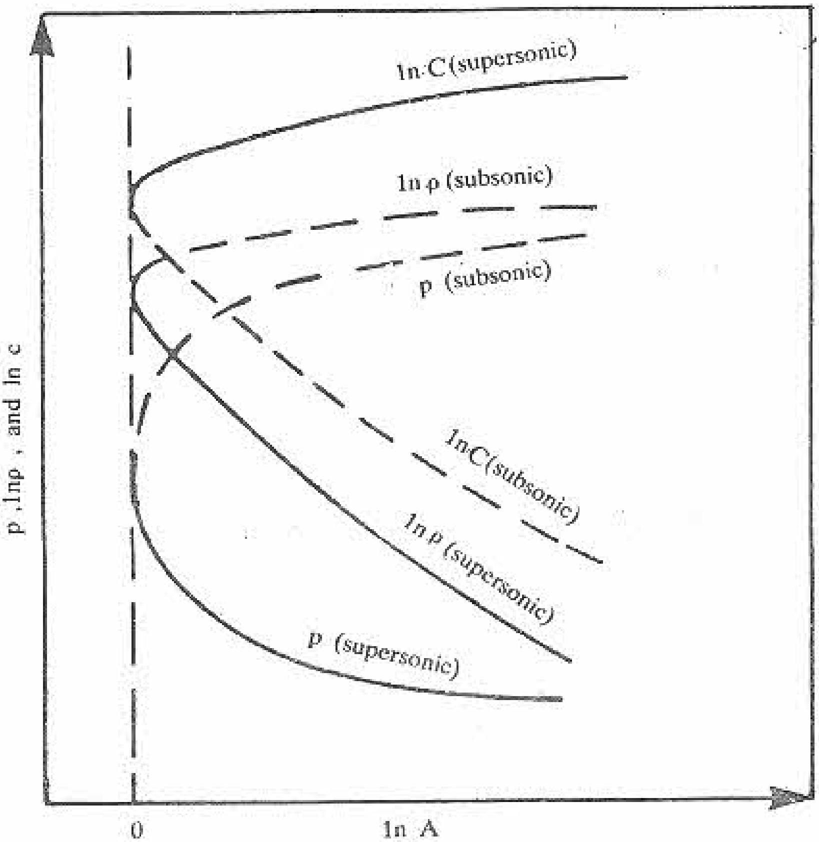


Fig. 5.4 Effects of area variation in compressible flow

For subsonic flow (M<1), Fig. (5.4) shows that;

i) The pressure increases with an area increase.
ii) The density increases with an area increase.
iii) The velocity decreases with an area increase.

For supersionic flow (M>1), Fig. (5.4) shows that;

i) The pressure decreases with an area increase.
ii) The density decreases with an area increase.
iii) The velocity increases with an area increase.

NOZZLES

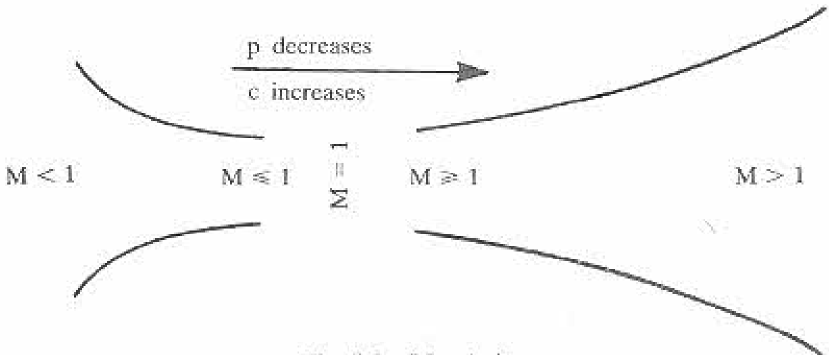


Fig. 5.5a (Nozzles)

DIFFUSERS:

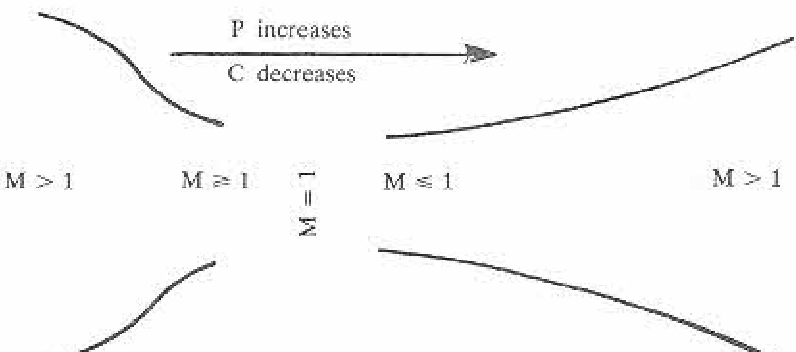


Fig. 5.5b (Diffusers)

Fig. 5.5 Effects of area change in subsonic and supersonic flow

For flow at sonic speeds ($M = 1$) the three equations (5.82), (5.84) and (5.86) show that;

i) $$\frac{dA}{dP} = 0 = \frac{dA}{d\rho} = \frac{dA}{dc} \qquad (5.87)$$

Thus the area may be either a maximum or a minimum. Fig. (5.4) shows that at $M = 1$ it must be a minimum.

ii)As the sonic conditions approached (from either side) the pressure changes (and also density changes) become very large for even a small area change.

iii) Combining equations (5.84) and (5.86) yields;

$$\frac{d\rho}{\rho} = - M^2 \quad \frac{dc}{c} \qquad (5.88)$$

Thus near sonic velocities the change in density and the change in velocity will compensate.

The use of the three equations (5.82), (5.84) and (5.86) in the design of nozzles and diffusers is quite significant.

A nozzle is a device in which we gain a velocity increase at the expense of pressure (pressure decrease). To accomplish this, in subsonic flow the nozzle area must decrease, and in supersonic flow the area must increase.

A diffuser is a device in which we gain a pressure increase at the expense of velocity (velocity decrease). This may be accomplished in subsonic flow by making the diffuserdiverges, and in supersonic flow by making the diffuser converges.

5 – 8 – Nozzle Design

Consider the nozzle shown in Fig. (5.6). The continuity equation for this nozzle shape states that

$$\rho C A = \rho_* C_* A_* \qquad (5.89)$$

i.e.

$$\frac{A}{A^*} = \frac{\rho_* C_*}{\rho c} = \frac{\rho_*}{\rho} \cdot \frac{C_*}{c} \qquad (5.90)$$

consider the ratio ρ_*/ρ which can be written using Eqs. (5.67) and (5.70), in the following form

$$\frac{\rho^*}{\rho} = \frac{\rho^*}{\rho_o} \cdot \frac{\rho_o}{\rho} = [\frac{2}{\gamma-1} (1 + \frac{\gamma-1}{2} M^2)]^{1/\gamma-1} \qquad (5.91)$$

Consider the ratio C_*/C and remember that

$$C_* = a_* = \sqrt{\gamma R T_*}$$

$$C = Ma = M\sqrt{\gamma R T}$$

therefore this ratio can be written as

$$\frac{C_*}{C} = \frac{a_*}{Ma} = \frac{1}{M}\sqrt{\frac{T_*}{T}} \tag{5.92}$$

Substituting into Eq. (2.92) from Eqs. (5.66) and (5.69) to get the ratio C_*/C as follows

$$\frac{C_*}{C} = \frac{1}{M}\sqrt{\frac{T_*}{T_o} \cdot \frac{T_o}{T}} = \frac{1}{M}\left[\frac{2}{\gamma+1}\left(1 + \frac{\gamma-1}{2} M^2\right)\right]^{1/2} \tag{5.93}$$

Substitution from Eqs. (5.91) and (5.93) into Eq. (5.90) gives

$$\frac{A}{A^*} = \frac{1}{M}\left[\frac{2}{\gamma+1}\left(1 + \frac{\gamma-1}{2} M^2\right)\right]^{(\gamma+1)/2(\gamma-1)} \tag{5.94}$$

This ratio is plotted in Fig. (5.7).

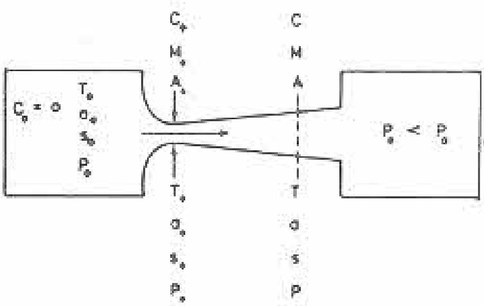


Fig. 5.6 Flow from a reservoir through a convergent-divergent nozzle

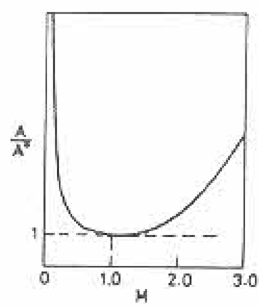


Fig. 5.7 Area ratio versus Mach number

To design a nozzle with the following conditions and requirements

- stagnation conditions P_o, ρ_o and T_o
- mass flow rate $\dot{m}$
- exit pressure P_e

the following steps are recommended

1 – Calculate the throat conditions from Eqs. (5.69), (5.70) and (5.71) as follows

$$T_* = T_o \left(\frac{2}{\gamma+1} \right) \quad (5.95)$$

$$\rho_* = \rho_o \left(\frac{2}{\gamma+1} \right)^{1/\gamma-1} \quad (5.96)$$

$$P_* = P_o \left(\frac{2}{\gamma+1} \right)^{\gamma/\gamma-1} \quad (5.97)$$

Also P_* can be calculated from equation of state, i.e.

$$P_* = \rho_* RT_*$$

2 – Calculate the velocity at the throat using the following relation

$$C_* = a_* = \sqrt{\gamma RT_*} \quad (5.98)$$

3 – The throat area is calculated from the continuity equation

$$\overset{o}{m} = \rho_* \; C_* \; A_* \quad (5.99)$$

4 – Use Eqs. (5.66), (5.67) and (5.68) and the exit pressure to calculated M_e, ρ_e and T_e are follows

$$\frac{P_e}{P_o} = \left(1 + \frac{\gamma-1}{2} M_e^2 \right)^{-\gamma/\gamma-1} \quad (5.100)$$

$$\frac{\rho_e}{\rho_o} = \left(1 + \frac{\gamma-1}{2} M_e^2 \right)^{-1/\gamma-1} \quad (5.101)$$

$$\frac{T_e}{T_o} = \left(1 + \frac{\gamma-1}{2} M_e^2 \right)^{-1} \quad (5.102)$$

ρ_e and T_e can be calculated from equation of state and relation of isentropic flow process.

5 – Calculate the velocity at exit from the relation

$$C_e = \sqrt{2 \frac{\gamma}{\gamma-1} \left(\frac{P_o}{\rho_o} - \frac{P_e}{\rho_e} \right)} \quad (5.103)$$

or from

$$C_e = M_e \; a_e = M_e \sqrt{\gamma RT_e} \quad (5.104)$$

6 – Find the exit area from the continuity consideration

$$\overset{\circ}{m} = \rho_e \, C_e \, A_e \tag{5.105}$$

Hence the dimensions of the nozzle at exit area fixed.

7 – If the uniformity of flow at exit is not a necessity, and the nozzle reaction is the goal, then one can state that the nozzle performance is not very sensitive to the geometry chosen. The geometry of the nozzle is generally chosen for reasons of ease of manufacture. The convergent part of the nozzle is often much shorter than the divergent part. A simple design is shown in Fig. (5.8).

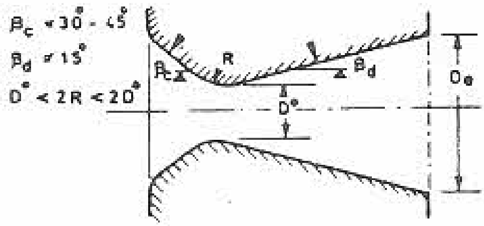


Fig. 5.8 Simple design of nozzle for which uniformity of flow at exit is not a necessity

8 – If the inlet radius is fixed from the combustion chamber considerations, then a recomended shape is shown in Fig. (5.9)

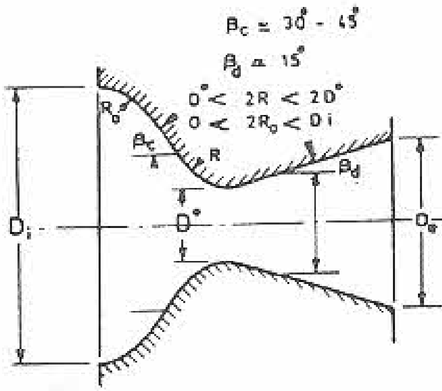


Fig. 5.9 Nozzle design for which the inlet radius is fixed

9 – For bigger pressure ratios P_o/P_e, the nozzle shall be quite long and heavy if the above proportions are used. It appears desirable to reduce the length by;

– first to expand the flow more quickly from the sonic conditions at the throat,

– then to turn or straighten it in the axial direction.

A sketch of a proposed configuration is given in Fig. (5.10)

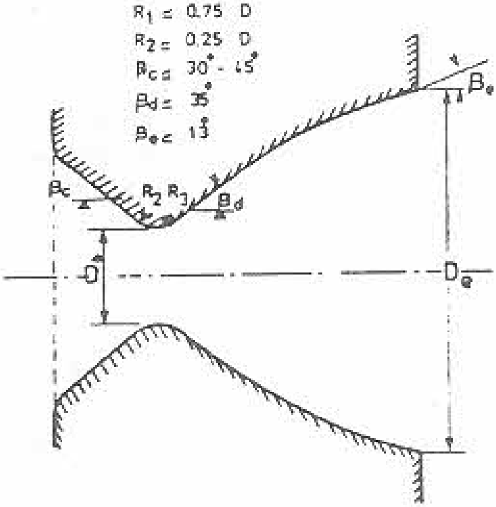
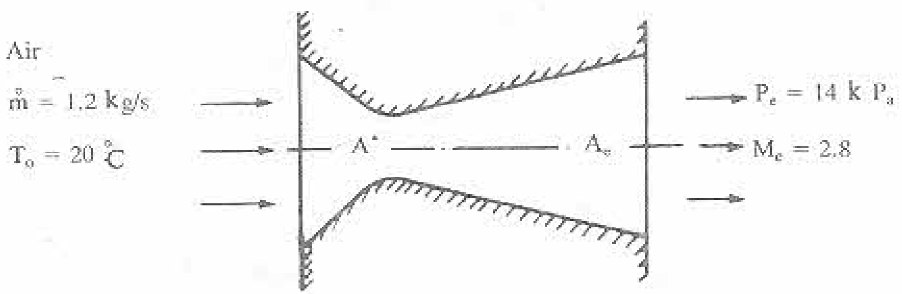


Fig. 5.10 Proposed configuration for nozzle with bigger pressure ratio P_o/P_e

EXAMPLE 5.4

Air is to flow through a convergent-divergent nozzle at 1.2 kg/s from a large reservoir in which the temperature is 20°c. At the nozzle exit the pressure is to be 14 k.Pa and the Mach number 2.8. Assuming isentropic flow, determine the throat and exit areas of the nozzle. Consider R = 287 J/kg. K, Y = 1.4.

Problem Description



Data of the Problem

* $\dot{m}$ = 1.2 kg/s

* T_o = 20°c = 293 °k
* P_e = 14 kPa = 14000 Pa
* M_e = 2.8
* R = 287 J/kg. K
* γ = 1.4

Requirements

* the throat area, A^*
* the exit area, Ae

Solution

The conservation of mass states that

$$\dot{m} = \rho^* A^* V^* = \rho_e A_e V_e \tag{I}$$

but

$$V^* = a^* \tag{II}$$

therefore from Eqs. (I) and (II)

$$A^* = \frac{\dot{m}}{\rho^* \; a^*} \tag{III}$$

$$A_e = \frac{\dot{m}}{\rho_e \; C_e} \tag{IV}$$

Use Eq. (5.41) to calculate P_o

$$\frac{P_o}{P_e} = [\,1 + \frac{\gamma-1}{2} \; M_e^2\,]^{\gamma/\gamma-1}$$

$$= [\,1 + 0.2 \; (2.8)^2\,]^{3.5}$$

$$= 27.14 \tag{V}$$

Substitute from Requirements by P_e to get

$$P_o = 27.24 \times 14 \times 10^3 = 3.8 \times 10^5 \; P_a$$

Get the stagnation density, ρ_o, from the equation of state as follows

$$\rho_o = \frac{P_o}{R\,T_o}$$

$$= \frac{3.8\times10^5}{287 \times 293} = 4.52 \quad kg/m^3 \qquad \text{(VI)}$$

Use Eq. (5.40) to calculate the density of exit flow, i.e.

$$\frac{\rho_e}{\rho_o} = [\,1 + \frac{\gamma-1}{2} M_e^2\,]^{-1/\gamma-1}$$

$$= [\,1 + 0.2\ (2.8)^2\,]^{-2.5}$$

$$= 0.0946 \qquad \text{(VII)}$$

Substitute from Eq. (VI) instead of ρ_o to get ρ_e, i.e.

$$\rho_e = 0.428 \quad kg/m^3 \qquad \text{(VIII)}$$

Get the exit velocity from the definition of Mach number Eq. (5.12), and the temperature-ratio as a function of Mach number, Eq. (5.39), as follows

$$V_e = M_e \cdot a_e \qquad \text{(IX)}$$

$$= M_e \cdot \sqrt{\gamma R T_e}$$

and

$$\frac{T_e}{T_o} = [\,1 + \frac{\gamma-1}{2} M_e^2\,]^{-1}$$

$$= [\,1 + 0.2(2.8)^2]^{-1}$$

$$= 0.3894$$

i.e.

$$T_e = 114.1 \quad K$$

Substitution into Eq. (VIII) gives

$$V_e = 2.8\ \sqrt{1.4 \times 287 \times 114.1} \qquad \text{(X)}$$

$$= 599.5$$

Now substitute from Eqs. (VIII) and (X) into Eq. (IV). to calculate A_e, i.e.

$$A_e = \frac{\dot{m}}{\rho_e V_e}$$

$$= \frac{1.2}{0.428 \times 599.5} = 4.68 \times 10^{-3} \ m^2 \qquad \text{(XI)}$$

To conclude A^*, see Eq. (III), it needs to calculate ρ^* and a^* as follows

getting ρ^*

Use Eq. (5.52), i.e.

$$\rho^* = \rho_o \left(\frac{2}{\gamma+1} \right)^{1/\gamma-1}$$

$$= 4.52 \left(\frac{2}{1.4+1} \right)^{1/0.4}$$

$$= 2.866 \ kg/m^3 \qquad \text{(XII)}$$

getting a^*

Use Eqs. (5.11) and (5.51) as follows

$$a^* = \sqrt{\gamma R T^*} \qquad \text{(XIII)}$$

and

$$\frac{T^*}{T_o} = \frac{2}{\gamma+1}$$

$$= \frac{2}{1.4+1} = 0.833 \qquad \text{(XIV)}$$

i.e.

$$T^* = 293 \times 0.833 = 244.17 \ ^\circ k \qquad \text{(XV)}$$

substitute from Eq. (XV) into Eq. (XIII) to get

$$a^* = \sqrt{1.4 \times 287 \times 244.17}$$

$$= 313.22 \ m/s \qquad \text{(XVI)}$$

Substitution from Eqs. (XII) and (XVI) into Eq. (III) gives

$$A^* = \frac{\overset{\circ}{m}}{\rho^* \; a^*}$$

$$= \frac{1.2}{2.866 \times 313.22}$$

$$= 1.337 \times 10^{-3} \quad m^2 \qquad \text{(XVII)}$$

As a check, from Eqs. (XI) and (XVI)

$$\frac{A_e}{A^*} = 3.5 \qquad \text{(XVIII)}$$

and also use Eq. (5.50) to get the same area ratio

$$\frac{A_e}{A^*} = \frac{1}{M_e} \left[\frac{5 + M_e^2}{6} \right]^3$$

$$= \frac{1}{2.8} \left[\frac{5 + (2.8)^2}{6} \right]^3$$

$$= 3.5 \qquad \text{(XIX)}$$

From Eqs. (XVIII) and (XIX), the answer is correct.

PROBLEMS ON CHAPTER FIVE

Problems on Sections 5 – 1 to 5 – 5

5.1. A gas that obeys the law $Pv^{\gamma} =$ constant flows along a pipe. Prove that,

$$C^2 + \frac{2\,\gamma}{\gamma - 1}\,\frac{p}{\rho} =$$

constant for this process where; C is the velocity of the gas, P and ρ are its pressure and mass density respectively, and γ is the specific heat ratio of the gas.

5.2. The velocity and temperature at a point in an isentropic flow of helium are 113 m/s and 95°C, respectively. Predict the temperature on the same streamline where the velocity is 190 m/s. What is the ratio between the pressure at the two points. Consider $C_p = 5.223$ kJ/kg. K for helium.

5.3. A fluid for which $Pv^{\gamma} =$ constant, flows through a thin pipe leading out of a large closed vessel in which the pressure is m times the atmospheric pressure P_a. Show that the speed of efflux is given by

$$C^2 = \frac{2\gamma}{\gamma-1} \frac{P_a}{\rho_a} [m^{\frac{\gamma-1}{\gamma}} - 1]$$

ρ_a *being the density of the fluid at atmospheric pressure* P_a.

5.4. Show that for air the stagnation temperature-rise in degree Kelvin is approximately five times the square of the speed in hundreds of meters per second, i.e. $\Delta T = 5\ (C/100)^2$. (Note that for air $\gamma = 1.4$ and $R = 287$ J/kg.K).

5.5. The speed of reentry vehicle at an altitude of 9 km was 1000 m/s. Calculate the Mach number and the stagnation temperature. Consider T = 229.5 K.

5.6. Calculate the Mach number of an object travelling at 350 m/s in an environment of air at 232 K. Estimate also the corresponding stagnation temperature.

5.7. An airplane flies at 950 km/h at an altitude where the air temperature is 224.3 K. Estimate the temperature near the stagnation point on the fuselage.

5.8. An airplane is flying at a speed of 215 m/s at an altitude of 500 m, where the temperature is 20°C. The plane climbs to 15 km, where the temperature is −56°C and levels off at a speed of 311 m/s. Calculate the Mach number of flight in both cases.

5.9. Air flows from a reservoir at 60°C, 700 kN/m^2. Assume isentropic flow, calculate the velocity, temperature, pressure and density at a section where M = 0.6.

5.10. If the difference between static and stagnation pressure in standard air (P = 101.3 kPa; T = 15°C) is 67 cm of mercury compute the air velocity assuming

a) the air is incompressible.

b) the air is compressible.

5.11. A pitot static tube is inserted into the test section of a subsonic wind tunnel. It indicates a static pressure of 72 kN/m^2 while the difference between stagnation and static pressure is shown as 14 cm of mercury. The barometric pressure is 75.2 cm of mercury and the stagnation temperature is 38°C. Calculate the Mach number and the air velocity.

5.12. Products of combustion leave the nozzle of a rocket engine with a Mach number of 4. The pressure at this point is 65kN/m^2. The specific heat ratio for the combustion products is 1.3. What is the nozzle inlet stagnation pressure for isentropic flow? What are the ratios of static to stagnation temperature and static to stagnation density?

5.13. Find the mass of air flow per second through 10 cm^2 of area if the air is supplied from a tank where the pressure is maintained at 600 kN/m^2 and the temperature is 100°C and the Mach number varies as below;

a) 0.5, b) 0.8, c) 1.0, d) 1.5, e) 2.0.

5.14. An aeroplane is flying at a speed of 185 m/s at a height where the air pressure is 30 kN/m^2 and the air temperature is 228 K.

What pressure will be developed between its pitot and static tubes? Give two answers, one neglecting compressibility and the other taking it into account.

Problems on Sections 5 – 6 to 5 – 8

5.15. An ideal diffuser has air entering it with a Mach number 0.70.The area ratio of this diffuser is 2.0. Inlet conditions are P = 3 kN/m^2 and T = 70°C. Find;

a) the stagnation pressure at the inlet section.

b) the stagnation temperature at the inlet section.

c) the exit Mach number.

d) the exit pressure.

e) the exit temperature.

f) the exit velocity.

5.16. Air with a Mach number of 3.0 flows through a pipe with an area of 0.015 m^2. The pressure is 100 kN/m^2 and the temperature is 50°C. Find the stagnation pressure, the stagnation temperature, the mass flow per unit area, the minimum area which will give this mass flow, and the velocity where the area is 0.012 m^2.

5.17. Air is released from a pressure vessel at a constant mass rate. The nozzle through which it is released is of a length" ℓ", and the pressure is required to fall linearly with distance along the nozzle. Find the relation that gives the variation of the crossectional area of the nozzle along its length" ℓ".

Consider air to be a perfect gas and the process to be reversible adiabetic.

5.18. Nitrogen in sonic flow at a 22-mm diameter throat section has a pressure of 45kN/m^2, T= -15°C. Determine the mass flow rate.

5.19. Nitrogen flows from a large tank, through a convergent nozzle of 4 cm tip diameter, into the atmosphere. The temperature in the tank is 87°C. Calculate pressure, velocity, temperature, and sonic velocity in the jet; and calculate the flowrate when the absolute tank pressure is (a) 215 kPa and (b) 180 kPa. The atmospheric pressure is 102.3 kPa. What is the lowest tank pressure that will produce sonic velocity in the jet? What is this velocity and What is the flowrate?

5.20. Air flows from the atmosphere into an evacuated tank thorugh a convergent nozzle of 35 mm tip diameter. If atmospheric pressure and temperature are 103.14 kPa and 25°C, respectively, what vacuum must be maintained in the tank to produce sonic velocity in the jet?

What is the flow rat? What is the flowrate when the vacuum is 250 mm of mercury?

5.21. Air (at 35°C and 750 kPa absolute) in a large tank flows into a 200 mm pipe, where it discharges to the atmosphere (102.1 kPa) through a convergent nozzle of 80 mm tip diameter. Calculate pressure, temperature, and velocity in the pipe.

5.22. The Mach number and pressure at the entry of a subsonic diffuser are 0.88 and 420 kPa, respectively. Determine the area ratio required and the pressure rise if the Mach number at the exit of diffuser is 0.23. Assume isentropic diffuser of air.

5.23. Air flow through a simple covergent nozzle ending in a throat of 250 mm^2. The entrance pressure and temperature are 1000 kPa and 38°C, and the nozzle discharges into a large tank at 600 kPa pressure. The entrance velocity is negligible

a) Find the flowrate.

b) What is the maximum discharge rate obtainable with the given supply pressure? What is the corresponding maximum receiver tank pressure?

5.24. Air at 100 kPa and 297 K enters a diffuser with a speed of 185 m/s. Estimate the maximum possible air pressure at the exit.

5.25. Air flows isentropically through a converging nozzle. At the entrance the Mach number is 0.32, the pressure is 630 kPa, the temperature is 61°C, and the nozzle area is 0.001 m^2. The exit Mach number is 0.82. Calculate at the exit the temperature, stagnation temperature, pressure, stagnation pressure, density, velocity, and nozzle area.

5.26. Air in a tank has a pressure 720 kPa, and a temperature 32°C. It discharges into the atmosphere through a converging nozzle. Assuming isentropic flow, determine the exit pressure, temperature, and area for a mass flow rate of 0.83 kg/s.

5.27. Air flows in a nozzle at 517 m/s, $\rho = 1.86$ kg/m^3 and $T = 293$°C. Is an increase or decrease in area required to decrease the flow velocity?

5.28. Air is supplied to a converging-diverging nozzle at negligible velocity, 700 kN/m^2 and 300°C. The nozzle discharges into atmosphere which is 100 kN/m^2. Assuming that the flow is ideal and for a rate of 1 kg/s calculate;

a) the exit Mach number.

b) the throat pressure.

c) the throat and exit areas.

d) the throat and exit velocities.

5.29. A supersonic nozzle is to be designed for airflow with $M = 3.4$ at the exit section, which is 22 cm in diameter and has a pressure of 7.5 kN/m^2 and temperature of −78°C. Calculate the throat area and the reservoir conditions.

5.30. Air flows isentropically through a converging-diverging nozzle. The exit Mach number is 1.78. If the stagnation temperature and the static pressure at the exit are 335 K and 100 kPa, calculate.

a) the stagnation conditions (pressure, temperature and density) at the inlet.

b) the exit temperature.

c) the area ratio between the exit and the throat.

5.31. A supersonic nozzle expands air from 2300 kPa stagnation pressure and 1100 K stagnation temperature to an exit pressure of 435 kPa; the exit area of the nozzle is 100 cm^2

Determine

a) throat area.

b) pressure and temperature at the throat.

c) temperature at exit.

d) exit velocity as fraction of the maximum attainable velocity.

e) mass flow rate.

5.32. Air flows isentropically from atmosphere (pressure 102.3 kPa and temperature 25°C) to a 400 mm square duct where the Mach number is 1.7. Calculate the static pressure, the velocity and the mass flow rate in the duct. What is the minimum cross-sectional area upstream of this section?

5.33. The exit section of a convergent-divergent nozzle is to be used for the test section of a supersonic wind tunnel. If the absolute pressure in the test section is to be 65 kPa, what pressure is required in the reservoir to produce a Mach number of 3 in the test section? For the air temperature to be −15°C in the test section, what temperature is required in the reservoir? What ratio of throat area to test section area is required to meet these conditions?

5.34. A convergent-divergent nozzle of 60 mm tip diameter discharges to the atmosphere (101.5 kPa) from a tank in which air is maintaned at an absolute pressure and temperature of 700 kPa and 40°C, respectively. What is the maximum mass flowrate which can occur through this nozzle? What throat diameter must be provided to produce this flowrate?

5.35. A one-dimensional diffuser is designed to reduce supersonic flow to subsonic flow. Find an expression for the area ratio between the entrance and the throat as function of Mach number M and the specific heat ratio γ.

CHAPTER SIX
MEASUREMENTS, UNITS, DIMENSIONS AND DIMENSIONAL ANALYSIS

6 – 1 – Measurements and Units

Physical measurement is essentially a process of comparison. The tools for this comparison are units. To measure the length of a road is to compare it with a basic unit (for example the kilometer). The length of two roads can be compared by reference to this basic unit.

Units are either fundamental or derived. A fundamental unit is arbitrary in size. Its historical development has been based on human convenience. Examples of the fundamental units are the units of length (for example the metre, the yard, the mile, the kilometre... etc.) and the units of mass (for example the kilogram, the pound, the ton the ounce, the grain,.. etc). Derived units are not selected arbitrarily, but are obtained by some definite process (see Appendix D) from the fundamental units, for example the unit of area is the square whose side is the unit of length.

Different fundamental and derived units have been adopted since the early stages of scientific developments. In 1960 (Gregorian) the General Conference of Weights and Measures recommended that the "system International d'unites" to be known as SI should be taken into use instead of existing systems (see Appendix D). The reason for the choice was its simplicity and promising universality. Since then, many countries have been progressively adopting the SI. It is expected to become the only system for weights and measures throughout the world.

6 – 2 – Physical Dimensions of a Quantity

Assume an established unit of length and take a length equal to L of this unit, that is L is the measure of that length in terms of the established unit. The consistent unit of area would be a square whose side is the established unit. Thus a square whose side is L has the measure of its area as L^2 of the consistent units of area. Now we express this by saying that areas measured in consistent units have the dimensional formula L^2

Dimensional formulae for other physical quantities can also be derived. Take for example a period of time containing T of some established unit of time, then a consistent unit of velocity has the measure L/T in terms of the original units of length and time. Thus the dimensional formula of velocity is LT^{-1}. Similarly if a unit of mass has the measure M, then a consistent unit of momentum has the dimensional formula MLT^{-1}.

Conventionally the dimensional formula is known as the expression for the physical dimensions of the quantity.

If the symbol /// is used to denote dimensional equality and if M,L and T stand for the measures of the fundamental units of mass, length and time then we have;

Mass /// M
Length /// L
Time /// T

It must be emphasized that /// does not indicate either numerical equality or physical entity, but merely identifying the physical quantity as regards the dimensional formula. Thus if we have;

$M^a L^b T^c$ /// $M^x L^y T^z$

it follows that

$a = x, b = y$ and $c = z$

Fundamental and derived SI units in science and engineering, their symbols, definitions and dimensional formulae are listed in Table (6.1).

6 – 3 – Significance of Physical Dimensions

To illustrate the use of physical dimensions in a simple manner let us consider the work "W" required to lift a certain body to a certain height "H". We may assume for a start that the work required to lift this body is a function of the mass density "ρ", the gravitational acceleration "g" and the height "H". Hence it may be stated that.

W /// $\rho^a g^b H^c$

where

W /// $ML^2 T^{-2}$
ρ /// ML^{-3}
g /// LT^{-2}
H /// L

Table 6.1 SI Units in Science and Engineering, Their Symbols, Definitions and Dimensional Formulae.

Basic SI Units

Quantity	Name of unit	Recommended unit symbol	Dimensional Formula
Length	metre	m	L
Mass	kilogram	kg	M
Time	second	s	T
Electric current	*ampere	A	ϕ
Temperature	*kelvin	K	θ
Luminous intensity	candela	cd	$\simeq$
Amount of substance	mole	mol	-

Supplementary Basic Units

Plane angle	radian	rad	Non-dimensional
Solid angle	steradian	sr	Non-dimensional

Applied mechanics, mechanical engineering

Quantity	SI unit	Symbol	Definition	Dimensional Formula
Force	newton	N	$kg\ m/s^2$	MLT^{-2}
Work, energy, quantity of heat	joule	J	Nm	$ML^2\ T^{-2}$
Power, heat flow rate	watt	W	J/s	$ML^2\ T^{-3}$
Moment of force	newton metre	-	N m	$ML^2\ T^{-2}$
Pressure, stress	pascal	Pa	N/m^2	$ML^{-1}\ T^{-2}$
Temperature (basic)	kelvin	K	-	
Surface tension	newtons per metre	-	N/m	MT^{-2}
Thermal Coefficient of linear expansion	reciprocal kelvin	-	K^{-1}	θ^{-1}
Heat-flux density irradiance	watt per square metre	-	W/m^2	MT^{-3}
Thermal conductivity	watt per metre kelvin	-	W/m K	$MLT^{-3}\ \theta^{-1}$
Coefficient of heat transfer	water per square meter kelvin	-	W/m^2K	$MT^{-3}\ \theta^{-1}$
Heat capacity	joules per kelvin	-	J/K	$ML^2\ T^{-2}\ \theta^{-1}$

Contd... Table 6.1

Applied mechanics, mechanical engineering

Quantity	SI unit	Symbol	Definition	Dimensional Formula
Specific heat capacity	joules per kilogram kelvin	-	J/kg K	$L^2 T^2 \theta^{-1}$
Entropy	joules per kelvin	-	J/K	$ML^2 T^{-2} \theta^{-1}$
Specific entropy	joules per kilogram kelvin	-	J/kg K	$L^2 T^{-2} \theta^{-1}$
Specific energy	joules per kilogram	-	J/kg	$L^2 T^{-2}$
Specific latent heat				
Viscosity (kinematic)	metre square per	-	m^2/s	$L^2 T^{-1}$
Viscosity (dynamic)	metre squared per	-	Pa s	$ML^{-1} T^{-1}$
Electric resistance	ohm		V/A	$ML^2 T^{-3} \phi^{-2}$
Electric charge	coulomb	C	AS	ϕT
Electric potential difference or voltage or e.m.f.	volt	V	W/A	$ML^2 T^{-3} \phi^{-1}$
Electric conductance	siemens	S	A/V	$\phi^2 M^{-1} L^{-2} T^3$
Electric capacitance	farad	F	A s/V	$\phi^2 M^{-1} L^{-2} T^4$
Luminance	candela per square metre (nit)	-	cd/m^2	$\approx L^{-2}$
Illumination	lux	IX	$1 m/m^2$	$\approx L^{-2}$
Luminous flux	lumen	lm	cd sr	$\approx L^{-2}$
Frequency	hertz	Hz	s^{-1}	T^{-1}
Electric field strength	volts per metre	-	V/m	$ML^{-3} T^{-3} \phi^{-1}$
Electric flux density	coulombs per square metre	-	C/m^2	ϕTL^{-2}

Magnetic units

Quantity	SI unit	Symbol	Definition	Dimensional Formula
Magnetic flux	weber	Wb	Vs	$ML^2 T^{-2} \phi^{-1}$
Inductance	henry	H	V s/A	$ML^2 T^{-2} \phi^{-2}$
Magnetic field strength	amperes per metre	-	A/m	ϕL^{-1}
Intensity of magnetization	amperes per metre	-	A/m	ϕL^{-1}
Magnetic flux density	tesla	T	Wb/m^2	$MT^{-2} \phi^{-1}$

therefore

$$ML^2 T^{-2} /// (ML^{-3})^a (LT^{-2})^b L^c$$

that is

$$ML^2 T^{-2} /// M^a L^{(-3a+b+c)} T^{(-2b)}$$

It follows that

$$a = 1, \; -3a + b + c = 2$$
$$\text{and} \; -2b = -2$$

that is

$$b = 1 \text{ and } -3 + 1 + c = 2$$

that is

$$c = 4$$

Thus

$$W /// \rho g H^4$$

or

$$W /// \rho g H H^3$$

The last dimensional equality indicates that our assumption that the work W is a function only of the three physical quantites ρ, g and H is not correct. A fourth physical quantity must be taken into consideration. The dimensional fqrmula of this fourth guantity must be like H^3 that is L^3. This conclusion indicates that the volume of the body should also be considered in the analysis.

6 – 4 – Non-dimensional Quantities

A non-dimensional quantity is such that its measure is independent of the choice of the fundamental units, where it is to be understood that consistent units are employed. The dimensional formula for such a quantity, therefore has all its indices zeroes. The obvious nondimensional quantities are those which are defined as the ratio of two physical quantities of the same kind. Examples of these are; the coefficient of solid friction which is the ratio of two forces, and the angle in radians which is the ratio of two lengths.

Non-dimensional quantities are useful particularly for the purpose of modelling and experimenting under different conditions.

6 – 5 – Physical Dimensions of Differential Coefficients

Let y be a single valued continuous function of x, where each of x and y represent a particular physical quantity, then an increment Δy in y corresponds to an increment Δx in x, and a differential coefficient can be given as; dy/dx. Since x and y represent certain physical quantities then it is essential that the increment Δy be of the same dimensional formula as y, and the increment Δx be of the same dimensional formula as x, i.e.

$$\Delta y \;///\; y \quad \text{and} \quad \Delta x \;///\; x$$

Hence it follows that;

$$dy/dx \;///\; \frac{y}{x}$$

Also, by mathematical definition we have;

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

From which it follows that;

$$\frac{d^2y}{dx^2} \;///\; \frac{\text{an increment of (dy/dx)}}{\text{an increment of x}} \tag{6.1}$$

Again each increment has the dimensional formula of the quantity it represents, i.e. the numerator of equation (6.1) possesses the dimensional formula of (dy/dx) which is known to be identical to that of (y/x), and the denomenator possesses a dimensional formula identical to that of x, hence it follows that:

$$\frac{d^2y}{dx^2} \;///\; \frac{y/x}{x} \;///\; \frac{y}{x^2} \tag{6.2}$$

Proceeding step by step in this way, we arrive to the conclusion that

$$\frac{d^ny}{dx^n} \;///\; \frac{y}{x^n} \tag{6.3}$$

EXAMPLE 6.1

Deduce the dimensional formula for velocity and acceleration by using the first and the second derivatives of the distance x with respect to time t.

Data of the problem.

* velocity

$$v = \frac{dx}{dt}$$

* acceleration

$$a = \frac{d^2x}{dt^2}$$

Requirements

* Deduce dimensional formulae for v and a.

solution

x /// L, t ///T

Velocity $v = \frac{dx}{dt}$ /// $\frac{x}{t}$ /// $\frac{L}{T}$

Acceleration $a = \frac{d^2x}{dt^2}$ /// $\frac{x}{t^2}$ /// $\frac{L}{T^2}$

---

6 – 6 – The Physical Dimensions of Integrals

By definition the integral

$$I = \int_a^b y\ dx$$

Stands for the sum of the products y. Δx between the limits x = a and x = b. Since x has the same dimensional formula, it follows that

I /// xy

Assume we have a double integral

$$J = \int\int s\ dx\ dy$$

This again stands for the sum of a product of the type s. dx. dy between some specified limits and y analogy to the above argument it follows that

J /// s x y

Similarly for the general multiple integral

$$M = \int\int \cdots\cdots \int s\ dz_1\ dz_2 \ldots d\ z_m$$

we derive

$$M \;///\; s\, z_1 \; z_2 \ldots z_m \tag{6.4}$$

6 – 7 – Dimensional Analysis and the π -Theorem

Consider the dimensional equality

$$y_1 \;///\; y_2^a \; y_3^b \; y_4^c$$

and assume that the basic dimensions of the physical quantities y_1, y_2, y_3 and y_4 are three; namely mass M, length L and time T. Three equations in the three unknowns; a, b and c can now be deduced and the numerical values of a, b and c can be found. Also one nondimensional quantity can be formulated which relates the four physical quantities y_1, y_2, y_3 and y_4.

If, however, we have five physical quantities y_1, y_2, y_3, y_4 and y_5 in a dimensional equality such as

$$y_1 \;///\; y_2^a \; y_3^b \; y_4^c \; y_5^d$$

while the basic dimensions are still three, that is M,L and T, then the number of deducible equations in the unknowns a,b,c and d remains fixed at 3, in which case the numerical value of each of these indices can not be found, but may be expressed in terms of one another. In this particular case three of the indices can be expressed in terms of the fourth. Also in this case two non-dimensional quantities relating the five physical properties may be formulated.

Now if the number of physical properties connected by the dimensional equality is 6, then we will have five indices but only three equations evaluating three of these indices in terms of the remaining two, but in this case three non-dimensional quantities can be formulated.

As a rule if the number of the physical properties related by a dimensional equality is "n" and the number of basic dimensions involved in this equality is "B" then;

i) The number of unknown indices is "n – 1"

ii) The number of deducible equation in the unknown indices is "B"

iii) The number of non-dimensional quantities that can be formulated is "n-B"

The above argument is the basis of the main method of dimensional analysis which is known as Thomson's theorem of numerics or Bunckingham's π-theorem. This theorem is stated as follows;

If a physical relationship can be represented mathematically, by expressing one variable y_1 in terms of a number of other independent variables $y_2, y_3, y_4 \ldots y_n$ such as

$$y_1 = f\,(y_2\,,\, y_3\,,\, y_4 \;---\; y_n) \tag{6.5}$$

or

$$f\,(y_1\,,\, y_2\,,\, y_3\,,\, y_4 \;---\; y_n) = 0 \tag{6.6}$$

then a simplified version of this relation may be obtained by a process of reducing the number of variables into a lesser number of non-dimensional quantities. This can be explained as follows;

i) Let a number "B" of the "n" variables $y_1, y_2, y_3 \ldots y_n$ be regarded as primary y's where "B" is equal to the number of basic dimensions (e.g. length L, mass M, time T and temperature θ) required to describe the physical quantities involved. Note that each basic dimension must appear at least once among the dimensions of the primary y's.

ii) The remaining "n-B" y's are now expressed as "n-B" dimensionless ratios known as "numerics" or π quantities using the primary y's for the purpose of forming these ratios. In other words each π quantity is formed from products or ratios of powers of y's.

iii) Statement (6.6) may now be replaced by

$$\phi\,(\pi_1,\ \pi_2,\ \pi_3, \ldots, \pi_{n-B}) = 0 \tag{6.7}$$

The meaning of this theorem is to be illustrated by the following.

6-7-1- Selective Choices Method For Establishing Dimensionless Parameters Representing Force of Drag on a Submerged Body

If we confine ourselves to flow systems or models which have geometrical similarity we can say that the force exerted by a fluid stream flowing past a submerged body depends on the physical properties of the fluid say; density ρ and viscosity μ, on the velocity c of the stream and on the linear scale or size ℓ of the body. Hence it may be assumed that force of drag F is a function of ρ, μ, ℓ and c so that;

$$F = f\,(\rho, \mu, \ell, c) \tag{6.8}$$

or

$$\phi\,(F, \rho, \mu, \ell, c) = 0 \tag{6.9}$$

In this case there are five y's, that is n = 5 having three dimensions M,L and T that is B = 3. In particular the y's are

F /// MLT^{-2} , ρ /// ML^{-3}

μ /// $ML^{-1}\ T^{-1}$, ℓ /// L

C /// LT^{-1}

The choice of the three dimensions M,L and T follows from the fact that we are concerned with purely mechanical phenomena in this example. However, if the drag is expected to be tremendously high resulting in substantial temperature rise then a fourth dimension θ representing temperature may be taken into account. In fact this fourth dimension becomes quite relevant in convection heat transfer.

Now various choices are possible for the primary y's and they can be considered in turn.

First Choice

Take ρ, ℓ, c, as primary y's we then require dimensionless ratios for F and μ. For π_1 let us try

$$F\ /\ \rho^{a}\ c^{b}\ \ell^{d} \tag{6.10}$$

If this quantity is to be dimensionless then we must have

$$F\ ///\ \rho^{a}\ c^{b}\ \ell^{d} \tag{6.11}$$

that is

$$M\ L\ T^{-2}\ ///\ (ML^{-3})^{a}\ (LT^{-1})^{b}\ L^{d} \tag{6.18}$$

that is

$$a = 1 \quad , \quad 1 = -3a + b + d$$
$$\text{and} \quad -2 = -b$$

that is

$$b = 2 \quad \text{and} \quad d = 2$$

Hence we obtain

$$\pi_1 = F/\ \rho c^{2}\ \ell^{2}$$

By a similar argument we can obtain

$$\pi_2 = \mu / \rho c \ell$$

The method of dimensional analysis therefore reduces the original statement (6.9) to the much simpler statement

$$\phi\left(\frac{F}{\rho c^2 \ell^2}, \frac{\mu}{\rho c \ell}\right) = 0 \qquad (6.12)$$

That is we have reduced a relation involving five variables to a relation involving only two variables. Statement (6.12) can equally be expressed as

$$\frac{F}{\rho c^2 \ell^2} = f_1\left(\frac{\rho c \ell}{\mu}\right) \qquad (6.13)$$

where

$$F/\rho c^2 \ell^2$$

is known as a force coefficient C_d, and $\rho c \ell/\mu$ is known as Reynolds number Re. Hence statement (6.13) can be put in the form

$$C_d = f_1(Re) \qquad (6.14)$$

<u>Second Choice</u>

This time take μ, ℓ and c as the primary y's. We then require dimensionless ratios for F and ρ. Using the same method as before we obtain

$$\pi_1 = \frac{F}{\mu c \ell} \quad \text{and} \quad \pi_2 = \frac{\rho c \ell}{\mu}$$

so that

$$\frac{F}{\mu c \ell} = f_2\left(\frac{\rho c \ell}{\mu}\right) \qquad (6.15)$$

where $F/\mu c \ell$ is an alternative form for force coefficient so that once again force coefficient is a function of Reynolds number.

Third Choice

Let us take ρ, μ and ℓ as the primary y's. We then require dimensionless ratios for F and C and we find that

$$\pi_1 = \frac{F\rho}{\mu^2} \quad \text{and} \quad \pi_2 = \frac{\rho c \ell}{\mu}$$

so that

$$\frac{F\rho}{\mu^2} = f_2 \left(\frac{\rho c \ell}{\mu} \right) \tag{6.16}$$

where $F\rho/\mu^2$ is yet another form of the force coefficient.

Fourth Choice

Taking ρ, μ and c as the primary y's, it is easily verified that the result is the same as in third choice above.

It should be mentioned here that statements (6.13), (6.14) and (6.15) are alternative forms of equation (6.9).

6 – 8 – The Comprehensive Approach to the Use of the π Theorem

The validity of the method of using the π -theorem may better be proved by the following alternative approach.

Considering the same problem of the force on a submerged body, we assume first of all that

$$F = f(\rho, \mu, \ell, c) \tag{6.17}$$

Let us suppose that this functional relationship can be expressed mathematically and more definitely as a series of i-terms, each formed from powers of ρ, μ, ℓ and c that is

$$F = \sum_i \alpha_i [\rho^{ai} \ \mu^{bi} \ \ell^{di} \ c^{ei}] \tag{6.18}$$

α_i being a dimensionless coefficient. Hence for each term we must have

$$F \ /// \ \rho^{ai} \ \mu^{bi} \ \ell^{di} \ c^{ei} \tag{6.19}$$

or

$$MLT^{-2} \ /// \ (ML^{-3})^{ai} \ (ML^{-1})^{bi} \ L^{di} \ (LT^{-1})^{ei} \tag{6.20}$$

From which we obtain

$$ai + bi = 1$$
$$-3ai - bi + di + ei = 1$$
$$-bi - ei = -2$$

We thus have three equations with four unknowns and we still have the following choices;

In the First Choice when expressing ai, di and ei in terms of bi we get

$$ai = 1 - bi$$
$$ei = 2 - bi$$
$$di = 2 - bi$$

and

$$F = \sum_i \alpha_i \left[\rho^{1-bi} \; \mu^{bi} \; \ell^{2-bi} \; c^{2-bi} \right]$$

$$= \sum_i \alpha_i \left[\rho \ell^2 \; c^2 \; \rho^{-bi} \; \mu^{bi} \; c^{-bi} \; \ell^{-bi} \right] \tag{6.21}$$

or

$$\frac{F}{\rho c^2 \; \ell^2} = \sum_i \alpha_i \left(\frac{\rho c \ell}{\mu} \right)^{-bi} \tag{6.22}$$

or

$$\frac{F}{\rho c^2 \; \ell^2} = f_1 \left(\frac{\rho c \ell}{\mu} \right) \tag{6.23}$$

i.e. the force coefficient is a function of Reynolds number.

In the Second and Third Choices we express bi, di and ei in terms of ai and ai, bi and di in terms of ei respectively and a similar argument like that in the first choice yields

$$\frac{F}{\mu c \; \ell} = \sum_i \alpha_i \left(\frac{\rho c \ell}{\mu} \right)^{ai}$$

or

$$\frac{F}{\mu c \ell} = f_2 \left(\frac{\rho c \ell}{\mu} \right) \tag{6.24}$$

and

$$\frac{F\rho}{\mu^2} = \sum_i \alpha_i \left(\frac{\rho c \ell}{\mu}\right)^{di}$$

or

$$\frac{F\rho}{\mu^2} = f_3\left(\frac{\rho c \ell}{\mu}\right) \qquad (6.25)$$

6 – 9 – **Physical Similarity**

Theoretical analysis seldom give a complete solution for an engineering problem. It is frequently necessary to turn to experimental results to complete the study.

Much of such experimental work may be obtained on the equipment subject the investigation or an exact duplicate of it, but a large part of this experimental work is carried out on scale models. Comparisons are usually made between the prototype (i.e. the full scale equipment) and the model. For such comparisons to be valid the sets of conditions associated with each of the prototype and model must be physically similar.

Similarity between the objects can be in one or many forms e.g.
Geometric: which is similarity in shape,
Kinematic: which is similarity in velocities and accelerations
Dynamic: which is similarity in forces.

Similarities in temperature distributions, electric fields and many other engineering phenomena are also possible.

A combination of two or more of the above similarities is known as physical similarity. An illustration of the use of such similarity is given in the following example.

EXAMPLE 6 – 2

The following example is intended to demonstrate both approaches of the π theorem that is, the Selective Choice Method section 6.7.1, and the Comprehensive approach section 6.8 and the use of physical similarity section 6.9.

The Problem

The power "P" required to pump a liquid through a pipeline depends on:
i) the length of the pipeline "ℓ"
ii) the diameter of the pipe used "D"
iii) the dynamic viscosity of the liquid "μ"

iv) the density of the liquid "ρ"
v) the velocity of the liquid in the pipeline "C"

Formulate non-dimensional quantities that relate P, ℓ, μ, ρ, C and D.

A pipeline in which oil is to flow at 2 m/s is to be simulated by a model pipeline in which water is to be circulated at 5 m/s. The density of the oil is 0.8 that of water and its viscosity is 10 times that of water. The power consumed for circulating water in the model is 10 kW. Find:

a – the geometrical similarity that dimensional analysis imposes between the prototype and model pipelines.

b – the power required to pump the oil in the pipeline.

Data of the Problem

* Pipeline with length ℓ and diameter D
* Fluid in the pipeline with dynamic viscosity μ density ρ and velocity C
* Power required to circulate fluid in the pipe line P
* Prototype pipeline has oil with density 0.8 that of water and viscosity 10 times that of water
* Oil flows at 2 m/s
* Water in the model flows at 5 m/s

Requirements

* Non-dimensional quantities relating P, ℓ, μ, ρ, C and D
* Geometrical similarity between model and prototype.
* Power required to pump oil in the pipe line

Solution

The variables we have and their dimensional formulae are as follows

$P /// ML^2 T^{-3}$
$\ell /// L$
$\mu /// ML^{-1} T^{-1}$
$\rho /// ML^{-3}$
$C /// LT^{-1}$
$D /// L$

Selective Choices Method

In the *first choice* we select P, ℓ, μ and ρ.
Thus

$$P \; /// \; \ell^{a} \; \mu^{b} \; \rho^{d}$$

that is

$$ML^2 T^{-3} \;///\; L^a (ML^{-1} T^{-1})^b (ML^{-3})^d$$

or

$$ML^2 T^{-3} \;///\; M^{b+d} L^{a-b-3d} T^{-b}$$

Equating indices we get

$$1 = b + d, \; 2 = a - b - 3d, \; -3 = -b$$

That is

$$b = 3, \; d = -2 \text{ and } a = -1$$

Hence

$$P \;///\; L^{-1} \mu^3 \rho^{-2}$$

Giving a non-dimensional parameter

$$\pi_1 = \frac{P \ell \rho^2}{\mu^3}$$

which may be called power coefficient

In the *second choice* we select μ, ρ, and C which in a similar manner as above yields a non-dimensional parameter

$$\pi_2 = \frac{\rho C \ell}{\mu}$$

known usually as Reynolds number.

In the *third choice* we select μ, ρ, C and D which yields another form of Reynolds number

$$\pi_3 = \frac{\rho CD}{\mu}$$

In the *fourth choice* we may select P, ρ, C and D which give another form of the power coefficient

$$\pi_4 = \frac{P}{\rho C^3 D^2}$$

A *fifth choice* may be suggested as ρ, C, D and ℓ thus

$$\rho \;///\; C^e \; D^f \; \ell^g$$

that is

$$ML^{-3}\,T^{0} \;///\; (LT^{-1})^e \; L^f \; L^g$$

when equating the indices of M this gives $1 = 0$ which is not true. Thus indicating that this choice cannot form a nondimensional parameter.

The Comprehensive Approach

Now following the comprehensive approach we get

$$P \;///\; \ell^a \; \mu^b \; \rho^d \; C^e \; D^f$$

that is

$$ML^2\,T^{-3} \;///\; L^a \, (ML^{-1}\,T^{-1})^b \, (ML^{-3})^d \, (LT^{-1})^e \, L^f$$

or

$$ML^2\,T^{-3} \;///\; M^{b+d} \; L^{a-b-3d+e+f} \; T^{-b-e}$$

Equating the indices on both sides, we get

$$1 = b + d \qquad (I)$$
$$2 = a - b - 3d + e + f \qquad (II)$$
$$-3 = -b - e \qquad (III)$$

giving

$$d = 1 - b$$
$$e = 3 - b$$
$$f = 2 - a - b$$

Thus we have

$$P \;///\; \ell^a \; \mu^b \; \rho^{1-b} \; C^{3-b} \; D^{2-a-b}$$

or

$$P \;///\; \rho C^3 \; D^2 \; (\frac{\ell}{D})^a \; (\frac{\mu}{\rho CD})^b$$

that is

$$\frac{P}{\rho C^3 D^2} \;///\; (\frac{\ell}{D})^a \; (\frac{\mu}{\rho CD})^b$$

that is

$$\frac{P}{\rho C^3 \; D^2} = F \; ((\frac{\ell}{D}) \; , \; (\frac{\rho CD}{\mu}))$$

Thus giving the three non-dimensional parameters,

$$\pi_4 = \frac{P}{\rho C^3 \; D^2}$$ Power coefficient

$$\pi_5 = \frac{\ell}{D}$$ which may be called scale coefficient

and

$$\pi_3 = \frac{\rho CD}{\mu}$$ Reynolds number

If we solve equations I, II and III in a different manner say by expressing a, b and d in terms of e and f we may obtain a power coefficient similar to π_1 as well as the other two coefficients π_3 and π_5.

However, the possible non-dimensional parameters are:

$$\pi_1 = \frac{P \cdot \rho^2}{\mu^3} \; , \; \pi_4 = \frac{P}{\rho C^3 \; D^2} \; , \; \pi_5 = \frac{\ell}{D} \text{ and } \pi_3 = \frac{\rho CD}{\mu}$$

Now, the only non-dimensional parameter that contains geometrical dimensions exclusively is

$$\pi_5 = \frac{\ell}{D}$$

This represents the geometrical similarity imposed by dimensional analysis on both the prototype and the modle. This physically means that the ratio of the length of the pipeline to its diameter has to be the same for the prototype and the model.

Physical similarity between the prototype and the model is still governed by the other three parameters π_1, π_4 and π_3. Using any two of these should be enough for evaluating the power requirement for the oil pipeline as follows:

Taking the two parameters

$$\pi_1 = \frac{P\ell\ \rho^2}{\mu^3} \quad \text{and} \quad \pi_3 = \frac{\rho\ C\ \ell}{\mu}$$

then we have

$$\pi_1 \text{ (model)} = \pi_1 \text{ (prototype)}$$

$$\frac{P_m\ \ell_m\ \rho_m^2}{\mu_m^3} = \frac{P_p\ \ell_p\ \rho_p^2}{\mu_p^3} \qquad \text{(IV)}$$

Also

$$\pi_3 \text{ (model)} = \pi_3 \text{ (prototype)}$$

that is

$$\frac{\rho_m\ C_m\ \ell_m}{\mu_m} = \frac{\rho_p\ C_p\ \ell_p}{\mu_p}$$

or

$$(\ell_p/\ell_m) = \frac{\rho_m}{\rho_p} \cdot \frac{C_m}{C_p}\ \frac{\mu_p}{\mu_m}$$

Substituting for (ℓ_p/ℓ_m) into IV we get

$$\frac{P_p}{P_m} = \frac{\rho_m}{\rho_p} \cdot \frac{C_m}{C_p}\ \frac{\mu_p}{\mu_m}$$

that is

$$\frac{P_p}{P_m} = \frac{1}{0.8} \times \frac{2}{5} \times 10^2 = 50$$

or

$$P_p = 50 \times 10 = 500 \text{ kW}$$

PROBLEMS ON CHAPTER SIX

6.1. Determine the dimensional formula and the unit in S.I. for each of the following quantities:

(a) volume flow rate
(b) angular acceleration
(c) velocity gradient
(d) pressure
(e) torque
(f) work
(g) shear stress
(h) power
(j) energy
(k) momentum
(l) mass moment of inertia.

6.2. Determine the dimensional formula and the unit in S.I. for each of the following:

(a) bulk modulus $K = -v \left(\frac{\partial p}{\partial v}\right)_T$

(b) compressibility $k = -\frac{1}{v}\left(\frac{\partial v}{\partial p}\right)_T$

(c) dynamic viscosity $\mu = \frac{\text{shear stress}}{\text{velocity gradient}}$

(d) kinematic viscosity $\nu = \frac{\text{dynamic viscosity}}{\text{density}}$

(e) specific enthalpy $h = \frac{\text{energy}}{\text{unit mass}}$

(f) $C_p = \left(\frac{\partial h}{\partial T}\right)_p$

(g) gas constant R

(h) thermal conductivity $k = \frac{dQ}{dt} \cdot \frac{dx}{dT} \cdot \frac{1}{A}$

(j) heat transfer coefficient

(k) surface tension $h_o = \frac{1}{A} \; \frac{dQ}{dt} \times \frac{1}{\Delta T}$

v = specific volume, p = pressure, T = temperature,
C_p = specific heat, at constant pressure, Q = heat quantity,
t = time, x = distance, A = area.

6.3. Construct one dimensionless quantity from each of the following groups:

(a) ν, c, ℓ (b) c, ℓ, g (c) ρ, μ, ℓ, c

ν = kinematic viscosity, c = velocity, ℓ = length,
g = gravitational acceleration, ρ = density,
μ = dynamic viscosity.

6.4. Check the dimensional homogeneity of the following equalities:

(a) $p = \frac{\int p\, dV}{V_2 - V_1}$

(b) $W = mRT \ln \frac{p_1}{p_2}$

(c) $a = c \frac{dc}{dx}$

(d) $C_p\, dT = d\,(u + pv)$

(e) $PV = \frac{1}{3} Nmc^2$ (N is a dimensionless number)

(f) $dp = \frac{2\ RT}{(V - b)^2}\ dV + \frac{R}{V - b}\ dT$ (b is a constant of the same dimensions as V).

p = pressure, V = volume, W = work, m = mass, R = gas constant, T = temperature, a = acceleration, c = velocity, x = distance, C_p = specific heat at constant pressure, u = internal energy, v = specific volume.

6.5. Use dimensional analysis to correlate between the different quantities specified in each of the following groups. Indicate incorrect assumptions if there are any. Also show your correlations in non-dimensional form.

(a) The velocity "C" of propagation of surface waves on a shallow liquid which is assumed to depend only on the depth of liquid "d", the density "ρ" and gravitational acceleration "g".

(b) The period of a pendulum "t" which is assumed to depend on the mass "m", the length "ℓ", the angle of swing "α" and gravitational acceleration "g".

(c) The velocity "a" of propagation of sound in a fluid which is assumed to depend on the density "ρ", viscosity "μ" and the bulk modulus "K"

(d) The volume flow rat "dV/dt" of fluid passing through a nozzle from a pressurised reservoir which is assumed to depend on density "ρ", reservoir pressure "p" and nozzle diameter "d".

6.6. The volume flow rate "dV/dt" of a fluid through a sharp edged orifice plate depends on the orifice diameter "d", the pressure difference across the orifice "Δp", the density "ρ", and the viscosity "μ". Show that:

$$\frac{dV}{dt} = d^2 \left(\frac{\Delta p}{\rho}\right)^{\frac{1}{2}} \phi \left(\frac{\mu}{d\rho^{\frac{1}{2}}\, p^{\frac{1}{2}}}\right)$$

6.7. The amplitude "A" of vibration of the system shown depends on:

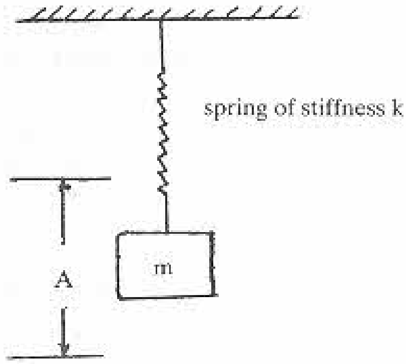
k = spring stiffness (force per unit length of expansion or compression of spring),

m = mass,

F_o = amplitude of disturbing force,

ω = frequency of disturbing force.

Use dimensional analysis to obtain the form of an equation relating "A" to these variables.



6.8. The power required to drive a propller depends on propeller diameter "D", mass density of the fluid "ρ", velocity of sound in the fluid "a", angular velocity of the propeller "N", Free-stream velocity "C", and viscosity of the fluid "μ". Deduce the dimensionless groups that characterise this problem and deduce a formula that gives the power in terms of these dimensionless group and any of the above parameters.

6.9. Pressure drop in pipe flow depends on pipe diameter "d", fluid velocity "C", fluid density "ρ" and dynamic viscosity of the fluid "μ". Find a dimensionless quantity that correlates the variable parameters of this flow.

A test is to be carried out to simulate the flow of oil by using water in a particular length of piping. At what velocity should water flow through the pipe in order to fulfill this similarity knowing that the density of the oil is 0.8 that of water, the dynamic viscosity of water is one third that of the oil, and the oil is to flow in the pipe at 25 m/s.

6.10. The drag force "F" on a subsonic aircraft depends on the scale of the craft "ℓ", the speed of the aircraft "C", the density of the fluid "ρ", and the viscosity of the fluid "μ". Formulate dimensionless quantities that relate the above variables.

An aircraft is to fly at a speed of 300 m/s at 10,000 m altitude where the temperature and pressure are $-45°$C and 30 kN/m^2 respectively. A 1/20 th-scale model is tested in a pressurized wind tunnel in which air is at 15°C. What pressure and velocity should be used in the tunnel if the drag force on the model is to be 1/10th of that on the prototype, (for air, take $p = \rho RT$ and $\mu \propto T^{\frac{3}{2}}$).

6.11. The maximum pitching moment produced by the water on the hull of a flying boat as it lands "M_{max}" depends on the following variables:

angle between the flight path and the horizontal "α"

angle defining atitude of aircraft "β"

mass of aircraft "m", length of hull "ℓ"

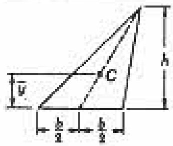
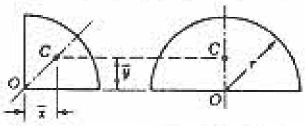
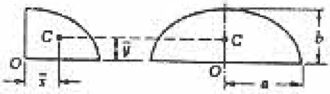
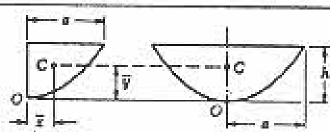
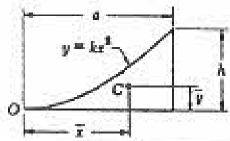
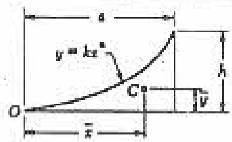
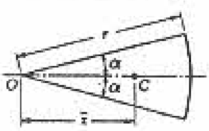
radius of gyration of aircraft about its axis of pitch "R"

water density "ρ" and gravitational acceleration "g".

Use dimensional analysis to obtain a form of equation for "M_{max}" in terms of these variables.

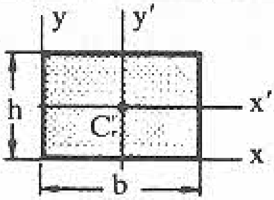
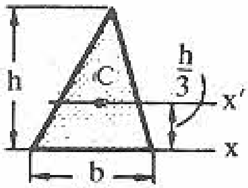
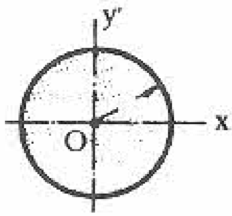
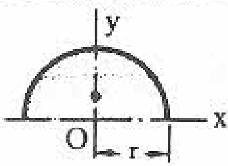
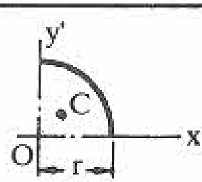
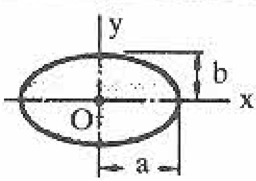
APPENDIX A

CENTROIDS OF COMMON SHAPES OF AREA

Shape		$\bar{x}$	$\bar{y}$	Area
Triangular area			$\frac{h}{3}$	$\frac{bh}{2}$
Quarter-circular area		$\frac{4r}{3\pi}$	$\frac{4r}{3\pi}$	$\frac{\pi r^2}{4}$
Semicircular area		0	$\frac{4r}{3\pi}$	$\frac{\pi r^2}{2}$
Quarter-elliptical area		$\frac{4a}{3\pi}$	$\frac{4b}{3\pi}$	$\frac{\pi ab}{4}$
Semielliptical area		0	$\frac{4b}{3\pi}$	$\frac{\pi ab}{2}$
Semiparabolic area		$\frac{3a}{8}$	$\frac{3h}{5}$	$\frac{2ah}{3}$
Parabolic area		0	$\frac{3h}{5}$	$\frac{4ah}{3}$
Parabolic spandrel		$\frac{3a}{4}$	$\frac{3h}{10}$	$\frac{ah}{3}$
General spandrel		$\frac{n+1}{n+2}a$	$\frac{n+1}{4n+2}h$	$\frac{ah}{n+1}$
Circular sector		$\frac{2r\sin\alpha}{3\alpha}$	0	αr^2

APPENDIX B

MOMENTS OF INERTIA OF COMMON GEOMETRIC SHAPES

Rectangle		$I_{x'} = \frac{1}{12} bh^3$ $I_{y'} = \frac{1}{12} b^3 h$ $I_x = \frac{1}{3} bh^3$ $I_y = \frac{1}{3} b^3 h$
Triangle		$I_{x'} = \frac{1}{36} bh^3$ $I_x = \frac{1}{12} bh^3$
Circle		$I_x = I_y = \frac{1}{4} \pi r^4$
Semicircle		$I_x = I_y = \frac{1}{8} \pi r^4$
Quarter circle		$I_x = I_y = \frac{1}{16} \pi r^4$
Ellipse		$I_x = \frac{1}{4} \pi ab^3$ $I_y = \frac{1}{4} \pi a^3 b$

APPENDIX C
THE PROCESS PATH FOR A GAS UNDERGOING A GENERAL QUASI-STATIC (REVERSIBLE) THERMODYNAMIC PROCESS

The Polytropic Reversible Process

Let us consider the application of the law of conservation of energy to a macroscopically small fixed amount of gas "Δm" in a fluid continuum undergoing a general quasi-static (reversible) thermodynamic process.

In its general form the law of conservation of energy states that:

$$(d\,Q_i - d\,Q_o) + (d\,W_i - d\,W_o) + (dm_i\,\varepsilon_i - dm_i\,\varepsilon_o)$$

$$= dE = d(m\,\varepsilon) \qquad (C.1)$$

When applied to the above quantity "Δm" and since $dm_i = 0$ and $dm_o = 0$, the above equation (6.1) reduces to:

$$(d\,Q_i - d\,Q_o) + (d\,W_i - d\,W_o) = \Delta m\,d\varepsilon \qquad (C.2)$$

We may also put, $Q_i = \Delta m\ q_i$ and $Q_o = \Delta m\ q_o$ where q_i and q_o are respectively heat added and heat rejected per unit mass of the system "Δm", so that,

$$d\,Q_i = \Delta m\,dq_i \text{ and } dQ_o = \Delta m\,dq_o$$

Also we may put:

$$W_i = \Delta m\,w_i \text{ and } W_o = \Delta m\,w_o$$

where w_i and w_o are respectively work done and work received per unit mass of the system so that:

$$d\,W_i = \Delta m\,d\,w_i \quad \text{and} \quad d\,W_o = \Delta m\,d\,w_o$$

Substituting for dQ_i, dQ_o, dW_i and dW_o from the above relations into equation (C.2) yields:

$$(\Delta m\,dq_i - \Delta m\,dq_o) + (\Delta m\,d\,w_i - \Delta m\,d\,w_o)$$

$$= \Delta m\,d\varepsilon \qquad (C.3)$$

so that:

$$d\,q_i - d\,q_o + d\,w_i - d\,w_o = d\,\varepsilon \qquad \text{(C.4)}$$

The quantity ε is also expressed as:

$$\varepsilon = (u + \frac{C^2}{2} + gz)$$

where no chemical energy is involved.

However, close examination of the different components of the quantity ε indicates that both kinetic energy and potential energy are completely interchangeable with work. In other words any gain or loss in either of these quantities will be on the expense of either work input or work output to the mass Δm. Even if such gain or loss is on the expense of the internal thermal energy "$\Delta m\, u$", kinetic energy and potential energy can still be classified as work and may be included in the terms dW_i and dW_o, so that the quantity ε is only equal to the specific internal thermal energy of the mass. Hence $\varepsilon = u$, or $d\,\varepsilon = du$, so that equation (C.4) can now be written as,

$$(d\,q_i - d\,q_o) + (d\,w_i - d\,w_o) = d\,u \qquad \text{(C.5)}$$

The specific quantities of heat-exchange q_i and q_o between the system and its surroundings may also be evaluated in terms of the temperature "T" of the mass Δm that is to say:

$$q_i = f_i(T) \quad \text{and} \quad q_o = f_o(T) \qquad \text{(C.6)}$$

so that:

$$d\,q_i = f_i'(T)\,dT \quad \text{and} \quad d\,q_o = f_o'(T)\,dT \qquad \text{(C.7)}$$

Now $f_i(T)$ and $f_o(T)$ would be the specific quantities of heat (i.e. heat per unit mass) that the mass Δm receives from and/or rejects to its surroundings for a change dT in its own temperature.

As shown before the quantity du can be given as:

$$du = (\frac{\partial u}{\partial T})\,dT = C_v \cdot dT \qquad \text{(C.7a)}$$

Now substituting the quantities(C.6), (C.7) and(C.7a) into equation (6.5) we get:

$$(f_i'(T)\,dT - f_o'(T)\,dT) + (p\,dv_i - p\,dv_o) = (\frac{\partial u}{\partial T})\,dT \qquad \text{(C.8)}$$

In most thermodynamic processes, while heat added and heat extracted from a system may be expressed separately as functions of the temperature of the system the work interaction can be expressed in terms of the pressure and volume of the system. However in the case of work being received by the system dv_i should be negative and in the case of work being done by the system dv_o is positive. Thus the quantity $pdv_i - pdv_o$ becomes $p(-dv_i - dv_o)$ or in general $-pdv$. Equation (C.8) therefore becomes:

$$-p\ dv = (\frac{\partial u}{\partial T})\ dT - f_i'\ (T)\ dT + f_o'\ (T)\ dT \qquad (C.9)$$

This equation gives the process path for a gas undergoing a general process in terms of its basic properties p,v and T.

For ideal gases we have:

$$pv = f(T) \text{ or } p = f(T)/v$$

From which into equation (C.9)we get:

$$-f(T)\ \frac{dv}{v} = \{\frac{\partial u}{\partial T} - f_i'\ (T) + f_o'\ (T)\}\ dT$$

or

$$-\frac{dv}{v} = \{(\frac{\partial u}{\partial T}) - f_i'\ (T) + f_o'(T)\}\ \frac{dT}{f(T)} \qquad (C.10)$$

Now assume that the gas undergoing this general process is a perfect gas whose equation of state is : $pv = RT$ then we have:

$$f(T) = RT$$

and by definition the specific heat at constant volume of this gas is constant so that:

$$(\frac{\partial u}{\partial T}) = C_v = \text{constant}$$

Assume also that the specific quantities of heat $f_i'(T)$ and $f_o'(T)$ are constants so that:

$$f_i'(T) = a \quad \text{and} \quad f_o'(T) = b$$

Now substituting for the quantities $f_i'(T)$, $f_o'(T)$, $\frac{\partial u}{\partial T}$ and $f(T)$ in equation (6.10) we get:

$$-\frac{dv}{v} = (C_v - a + b)\ \frac{dT}{RT} = (\frac{C_v - a + b}{R})\ \frac{dT}{T} \qquad (C.11)$$

that is

$$(\frac{C_v - a + b}{R})\ \frac{dT}{T} + \frac{dv}{v} = 0 \qquad (C.12)$$

Integrating this equation gives:

$$v\,T^{\left(\frac{C_v - a + b}{R}\right)} = \text{Constant} \tag{C.13}$$

$$T\,v^{\left(\frac{R}{C_v - a + b}\right)} = \text{Constant} \tag{C.14}$$

or

Employing the equation of state for a perfect gas, that is pv = RT, into the above equation we get:

$$p\,v^{\left(1 + \frac{R}{C_v - a + b}\right)} = \text{Constant} \tag{C.15}$$

This relation may be stated as:

$$p\,v^n = \text{constant} \tag{C.16}$$

where

$$n = 1 + \frac{R}{C_v - a + b} \tag{C.17}$$

Putting

$$R = C_v\,(\gamma - 1) \tag{C.18}$$

we get

$$n = 1 + \frac{R}{C_v - a + b} = \frac{\gamma\,C_v - a + b}{C_v - a + b} \tag{C.19}$$

A general quasi-static thermodynamic process in which the three fundamental properties p, v, and T of a gas change, is known as a polytropic process. If such a process is undergone by a perfect gas that exchanges heat with its surroundings at a constant rate (" $-a + b$ " energy units, per unit mass per degree temperature) all through, then this process is governed by equation (C.16)

Equation(C.19), however, indicates that "n" may have any value from $-\infty$ to $+\infty$ as follows:

If $(-a + b) = 0$ then $n = \gamma$
If $(-a + b) > 0$ then $\gamma > n > 1$
If $(-a + b) < 0$ then $-\infty < n < \infty$

These conclusions are better illustrated by Fig. (C.1)

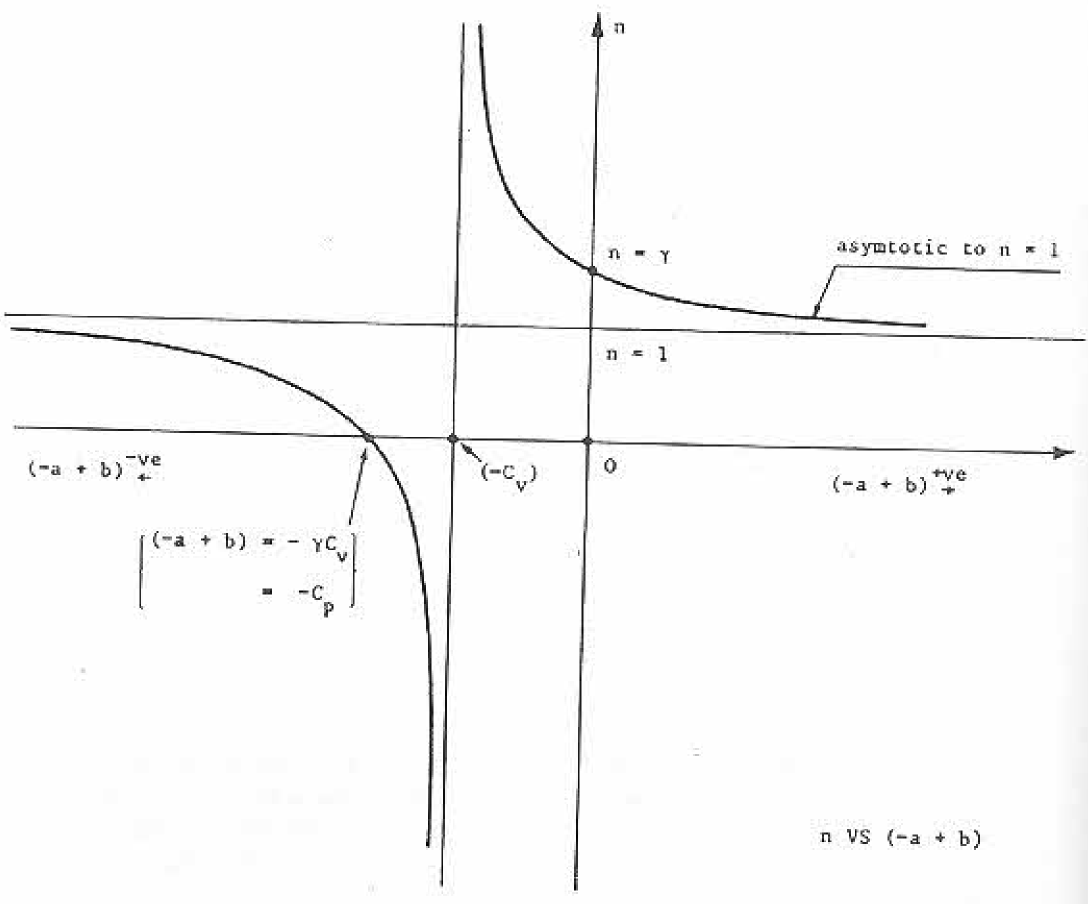


Fig. (C.1)

Now combining both relations pv^n = Constant and pv = RT for a perfect gas we get:

$$p_1 v_1^n = p_2 v_2^n \qquad \text{i.e.} \quad \frac{p_1}{p_2} = \left(\frac{v_2}{v_1}\right)^n$$

$$\frac{p_1 v_1}{T_1} = \frac{p_2 v_2}{T_2} \qquad \text{i.e.} \quad \frac{p_1}{p_2} = \frac{v_2}{v_1} \frac{T_1}{T_2}$$

so that:

$$\frac{v_2}{v_1} \frac{T_1}{T_2} = \left(\frac{v_2}{v_1}\right)^n$$

that is:

$$T_1 v_1^{n-1} = T_2 v_2^{n-1} \quad \text{or} \quad T v^{n-1} = \text{constant} \tag{C.20}$$

Also putting $v = \frac{RT}{p}$ in the above relation we get:

$$T_1 p_1^{\frac{1-n}{n}} = T_2 p_2^{\frac{1-n}{n}} = T p^{\frac{1-n}{n}} = \text{constant} \tag{C.21}$$

The relations governing the behaviour of a perfect gas undergoing a general quasistatic process (polytropic reversible) may now be summarised as follows:

pv^n = const $\qquad$ $p \rho^{-n}$ = const

$T v^{n-1}$ = const $\qquad$ $T \rho^{1-n}$ = const

$T p^{\frac{1-n}{n}}$ = const

$p v$ = RT

Constant Pressure Process

In a constant pressure process undergone by a perfect gas we consider the two relations:

$$T\,v^{n-1} = \text{const} \quad \text{and} \quad p\,v = RT$$

from which we get:

$$\frac{T_1}{T_2} = \left(\frac{v_2}{v_1}\right)^{n-1} \quad \text{and} \quad \frac{T_1}{T_2} = \frac{v_1}{v_2}$$

i.e.

$$\left(\frac{v_2}{v_1}\right)^{n-1} = \left(\frac{v_2}{v_1}\right)^{-1}$$

i.e.

$$n - 1 = -1 \quad \text{or} \quad n = 0$$

i.e.

$$\frac{\gamma\,C_v - a + b}{C_v - a + b} = 0$$

i.e.

$$\gamma\,C_v = (a - b) \quad \text{or} \quad C_p = (a - b)$$

Hence the net thermal energy added to the system per unit mass per one degree change in its temperature, (i.e. the quantity "a – b") must be equal in magnitude to the specific heat at constant pressure "C_p" of the gas.

Isothermal Process

In an isothermal process undergone by a perfect gas we consider the two relations,

$$pv^n = \text{constant and } pv = RT$$

from which we get:

$$\frac{p_1}{p_2} = \left(\frac{v_2}{v_1}\right)^{n} \quad \text{and} \quad \frac{p_1}{p_2} = \frac{v_2}{v_1}$$

or,

$$\left(\frac{v_2}{v_1}\right)^n = \frac{v_2}{v_1}$$

i.e.,

$$n = 1$$

or,

$$\frac{\gamma C_v - a + b}{C_v - a + b} = 1$$

or,

$$\gamma C_v - a + b = C_v - a + b$$

that is,

$$\gamma = 1 \quad \text{or} \quad C_v = \infty$$

in which case the relation fails to give sensible information. This is natural because the rate of exchange of heat between the system and its surroundings is not a function of temperature, while "a" and "b" are originally expressed as rates of heat interaction per degree temperature.

Constant Volume Process

In a constant volume process undergone by a perfect gas we consider the two relations:

$$T\, p^{\frac{1-n}{n}} = \text{constant} \quad \text{and} \quad p\, v = RT$$

which give:

$$\frac{T_2}{T_1} = \left(\frac{p_1}{p_2}\right)^{\frac{1-n}{n}} \quad \text{and} \quad \frac{T_2}{T_1} = \frac{p_2}{p_1}$$

i.e.

$$\left(\frac{p_1}{p_2}\right)^{-1} = \left(\frac{p_1}{p_2}\right)^{\frac{1-n}{n}}$$

that is:

$$\frac{1-n}{n} = -1 \quad \text{or} \quad 1-n = -n$$

That is :

$$1 = 0 \quad \text{(which cannot be true)} \quad \text{or } n = \infty$$

so that

$$\frac{\gamma\, C_v - a + b}{C_v - a + b} = \infty$$

that is

$$C_v - a + b = 0$$

or

$$C_v = (a - b)$$

which indicates that the net thermal energy added to the system per unit mass per degree change in its temperature must be equal to the specific heat at constant volume "C_v" of the gas.

Adiabatic Process

If the process undergone by the perfect gas is adiabatic (i.e. no heat exchanged between the gas and its surroundings) then we have

$$-a + b = 0$$

Hence the law followed by a perfect gas undergoing an adiabatic quasi-static process is:

$$p\, v^{\gamma} = \text{const}$$

Of course for this process we have

$$p\, v^{\gamma} = \text{const}, \qquad p\, \rho^{-\gamma} = \text{const}$$

$$T\, v^{\gamma-1} = \text{const}, \qquad T\, \rho^{1-\gamma} = \text{const}$$

$$T\, p^{\frac{1-\gamma}{\gamma}} = \text{const}, \qquad \text{and also,}$$

$$p\, v = RT$$

Different Processes on the p – v Diagram

Pressure-volume relationships of several quasi-static processes corresponding to different values of "n" are compared in Fig. (C.2).

Starting at point b expansion and compression are curves located in the lower right and upper left quadrents.

Processes that exhibit negative values of n are not commonly encountered in practice. Nevertheless these processes are possible, and they imply a simultaneous decrease in volume and pressure.

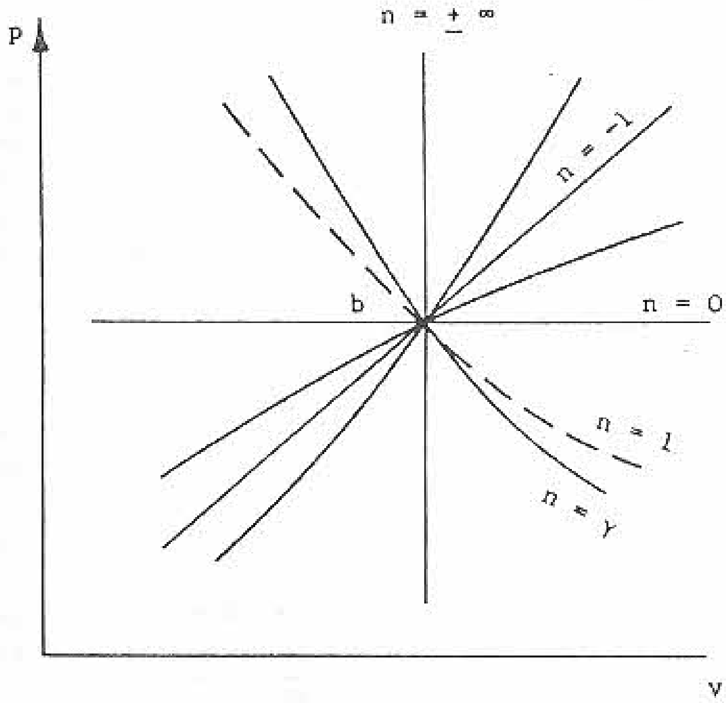


Fig. (C.2)

APPENDIX D
BASIC SI UNITS

No	Quantity	Name of unit	Recommended unit symbol
1	Length	metre	m
2	Mass	kilogram	kg
3	Time	second	s
4	Electric current	*ampere	A
5	Temperature	*kelvin	K
6	Luminous intensity	candela	cd
7	Amount of substance	mole	mol

Standard Multiples and Sub-multiples

Multiplication factor		Pre-fix	Sym-bol
One million million(billion)	1 000 000 000 000 $=10^{12}$	tera	T
One thousand million	1 000 000 000 $=10^{9}$	giga	G
One million	1 000 000 $=10^{6}$	mega	M
One thousand	1 000 $=10^{3}$	kilo	k
One hundred	100 $=10^{2}$	hecto	h*
Ten	10 $=10^{1}$	deca	da*
Unity	1 $=10^{0}$	-	-
One tenth	0.1 $=10^{-1}$	deci	d*
One hundredth	0.01 $=10^{-2}$	centi	c*
One thousandth	0.001 $=10^{-3}$	milli	m
One million	0.000 001 $=10^{-6}$	micro	μ
One thousand millionth	0.000 000 001 $=10^{-9}$	nano	n
One million millionth	0.000 000 000 001 $=10^{-12}$	pico	p
One thousand million millionth	0.000 000 000 000 001 $=10^{-15}$	femto	f
One million million millionth	0.000 000 000 000 000 001 $=10^{-18}$	atto	a

*It is suggested that all SI units be expressed in "preferred standard form" in which the multiplier is 10^{3n} where n is a positive or negative whole number. Consequently the use of hecto, deca, deci and centi is to be avoided wherever possible.

APPENDIX E
PARTIAL DERIVATIVES AND TOTAL DIFFERENTIALS

Consider $f = f(x,y)$ where x and y are independent variables. If y is held constant, f becomes a function of x alone and its drivetive may be determined as if it is a function of one variable. This is denoted by $\partial f/\partial x$ and is called the partial derivative of f with respect to x. Similarly if x is held constant, f becomes a function of y alone and $\partial f/\partial y$ is called the partial derivative of f with respect to y. These partial derivatives are defined by

$$\frac{\partial f}{\partial x} = \frac{\partial f(x,y)}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x, y) - f(x,y)}{\Delta x} \quad \text{(E.1)}$$

$$\frac{\partial f}{\partial y} = \frac{\partial f(x,y)}{\partial y} = \lim_{\Delta y \to 0} \frac{f(x, y+\Delta y) - f(x,y)}{\Delta y} \quad \text{(E.2)}$$

This is illulstrated as follows. If, for example,

$$f = x^3 + x^3 y^3 + 3y$$

then

$$\frac{\partial f}{\partial x} = 3x^2 + 3x^2 y^3$$

$$\frac{\partial f}{\partial y} = 3x^3 y^2 + 3$$

Also, if

$$f = \sin (ax^2 + by^2)$$

then

$$\frac{\partial f}{\partial x} = 2ax\cos (ax^2 + by^2)$$

$$\frac{\partial f}{\partial y} = 2by \cos (ax^2 + by^2)$$

In each case the differentiation is carried out exactly as for a function of one independent variable with the other independent variable considered as a constant.

Now, if x and y have infinitesimal changes Δx and Δy respectively the change in the function of becomes

$$\Delta f = f(x+\Delta x,\ y+\Delta y) - f(x,y) \tag{E.3}$$

where Δx and Δy may approach zero in any manner. Under this condition if Δf approaches zero regardless of the way in which Δx and Δy approach zero, then f = f(x,y) is called a continuous function of x and y. It will be assumed that f(x,y) is continuous and also that $\partial f/\partial x$ and $\partial f/\partial y$ are continuous. Equation (E.3) can be rewritten as follows

$$\begin{aligned}\Delta f &= f(x+ x,y+\Delta y) - f(x,y+\Delta y) \\ &\quad +f(x,y+\Delta y) - f(x,y) \\ &= \frac{f(x+\Delta x,y+\Delta y) - f(x,y+\Delta y)}{\Delta x} \quad \Delta x \\ &\quad + \frac{f(x,y+\Delta y) - f(x,y)}{\Delta y} \quad \Delta y\end{aligned} \tag{E.4}$$

On the limit when Δx, Δy and Δf tend to zeros, the above equation becomes

$$df = \frac{\partial f}{\partial x}\ dx + \frac{\partial f}{\partial y}\ dy \tag{E.5}$$

In general, if

$$f = f\ (x,y,z,t) \tag{E.6}$$

then

$$df = \frac{\partial f}{\partial x}\ dx + \frac{\partial f}{\partial y}\ dy + \frac{\partial f}{\partial z}\ dz + \frac{\partial f}{\partial t}\ dt \tag{E.7}$$

APPENDIX F

ENGINEERING DATA

Table F.1 Important constants

Constant	Value
Gas constant for air, R_{air}	287 J/kg C
Density of water at normal temperature and pressure, ρ_w	1000 kg/m^3
Density of mercury at normal temperature and pressure, ρ_{Hg}	$13.6x10^3$ kg/m^3
Standard acceleration of gravity (at sea level and latitude 45 degree), g	9.80665 m/s

Table F.2 Properties of saturated water (Taken from Simonson, 1975, with some modifications)

t ($^{\circ}C$)	ρ (kg/m^3)	C_p (J/kg deg C)	μ ($N.S/m^2$)	k (W/m deg C)	α (m^2/s)	Pr	β (1/deg K)
0	1002	4218	1.794×10^{-3}	0.552	13.1×10^{-8}	13.6	
20	1001	4182	1.011	0.597	14.3	7.02	0.18×10^{-3}
40	994.6	4178	0.654	0.628	15.1	4.34	
60	985.4	4184	0.470	0.651	15.5	3.02	
80	974.1	4196	0.355	0.668	16.4	2.22	
100	960.6	4216	0.282	0.680	16.8	1.74	
120	945.3	4250	0.233	0.685	17.1	1.446	
140	928.3	4283	0.199	0.684	17.2	1.241	
160	909.7	4342	0.172	0.680	17.3	1.099	
180	889.0	4417	0.154	0.675	17.2	1.004	
200	866.7	4505	0.139	0.665	17.1	0.937	
220	842.4	4610	0.126	0.653	16.8	0.891	
240	815.7	4756	0.117	0.635	16.4	0.871	
260	785.9	4949	0.108	0.611	15.6	0.874	
280	752.5	5208	0.102	0.580	14.8	0.910	
300	714.3	5728	0.0964	0.540	13.2	1.019	

Key to the table

t = temperature

ρ = density

C_p = specific heat

μ = kinematic viscasity

k = thermal conductivity

α = thermal diffusivity

P_r = Pandtl number

β = coefficient of thermal expansion

Table F.4 Density of some common liquids at standard atmospheric pressure

Liquid	Temp. °C	Density kg/m^3
Benzene	20	876.2
Crude oil	20	855.6
Ethyl alcohol	20	788.6
Freon-12	15.6	1345.2
Gasoline	20	680.3
Glycerin	20	1257.6
Hydrogen	-257.2	73.7
Mercury	10	13571
	20	13546
Water	20	998.2

Table F.5 Properties of some gases at room temperature

Gas	γ	R	C_p
Air	1.40	287.1	1 005
Helium	1.66	2077	5 224
Hydrogen	1.40	4124	14 434
Methane	1.31	518	2 190
Xenon	1.66	63.3	159

key:

γ ratio of specific heat at constant pressure to that at constant volume

R gas constant, J/kg.K

C_p specific heat at constant pressure, J/kg.K

Table F.6 Values of the bulk modulus and the vapor pressure of some common liquids at standard atmospheric pressure

Liquid	Temperature °C	Bulk Moldulus k Pa	Vapor Pressure k Pa
Benzene	20	1 034 250	10.0
Ethyl alcohol	20	1 206 625	5.86
Glycerin	20	4 343 850	0.000 014
Hydrogen	- 257.2	—	21.4
Mercury	15.6	26 201 000	0.000 17
Oxygen	- 195.6	—	21.4
Water	20	2 068 500	2.34

Table F.7 Surface tension of some liquids in contact with air, water, or their own vapours

SUBSTANCE	SURFACE TENSION (N/m)
Benzene-air	0.029
Carbon tetrachloride-air	0.027
Water-air	0.073
Mercury-air	0.435
Ethyl alcohol-air	0.022
Glycerin-air	0.063
Benzene-water	0.035
Carbon tetrachloride-water	0.045
Mercury-water	0.375
Benzene vapor	0.029
Carbon tetrachloride vapor	0.027
Water vapor	0.073

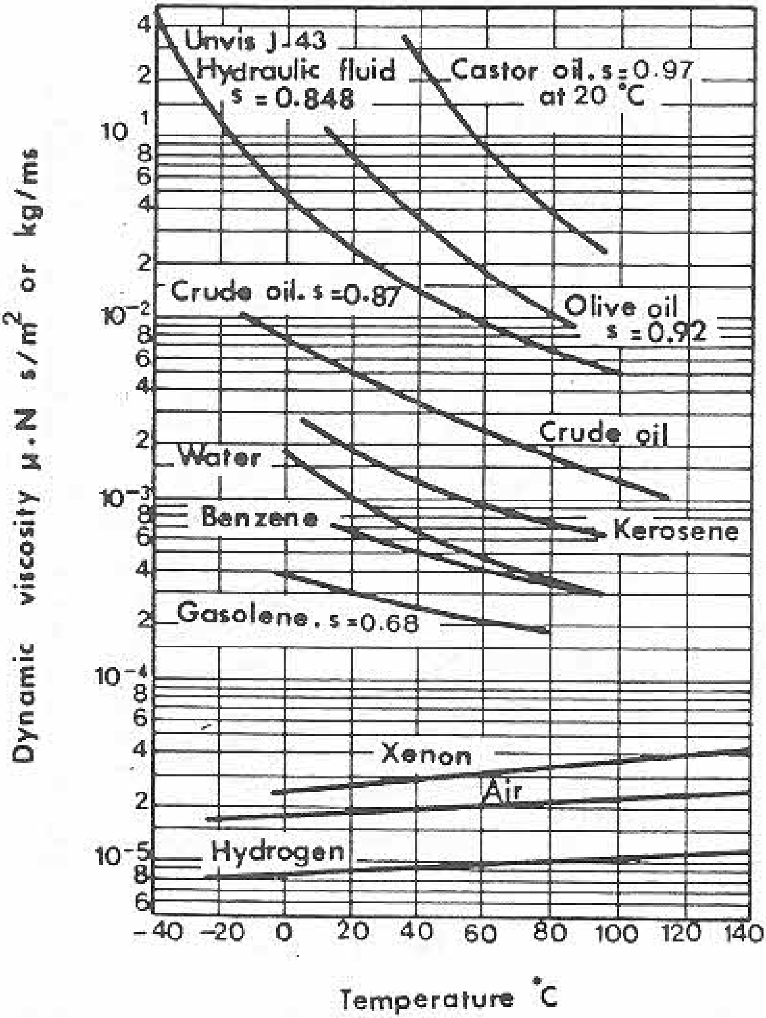


Fig. F.1 Dynamic viscosity of some liquids and gases. Values of specific gravity apply at about 20°C

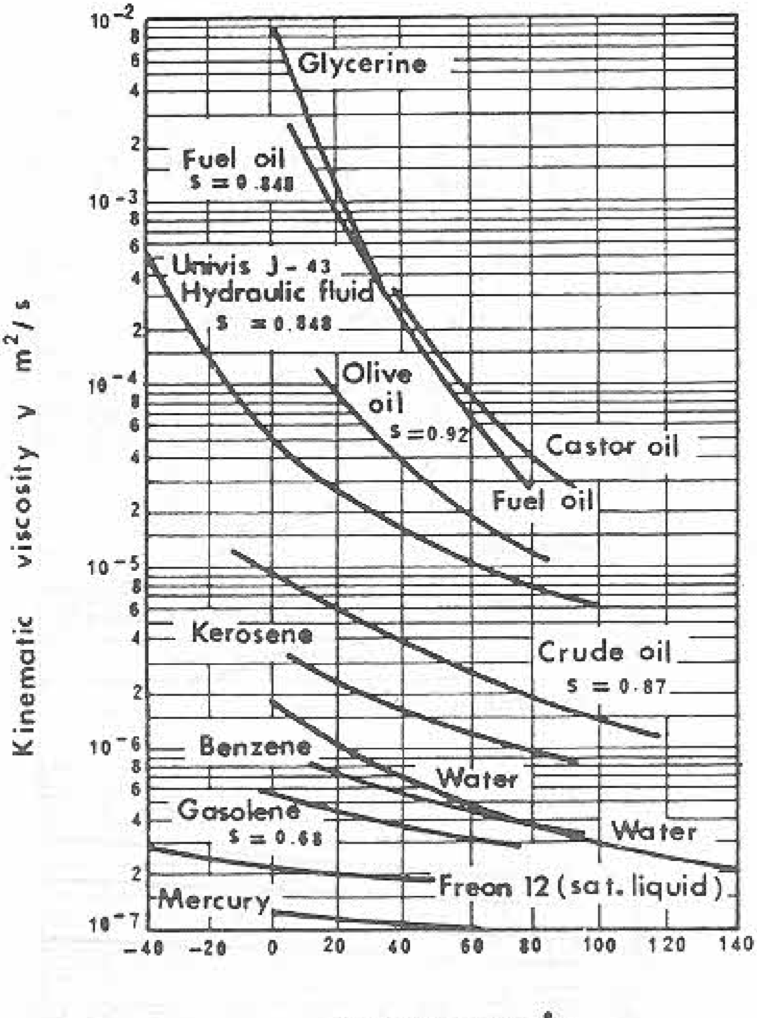


Fig. F.2 Kinematic viscosity of some liquids of specific gravity apply at about 20°C

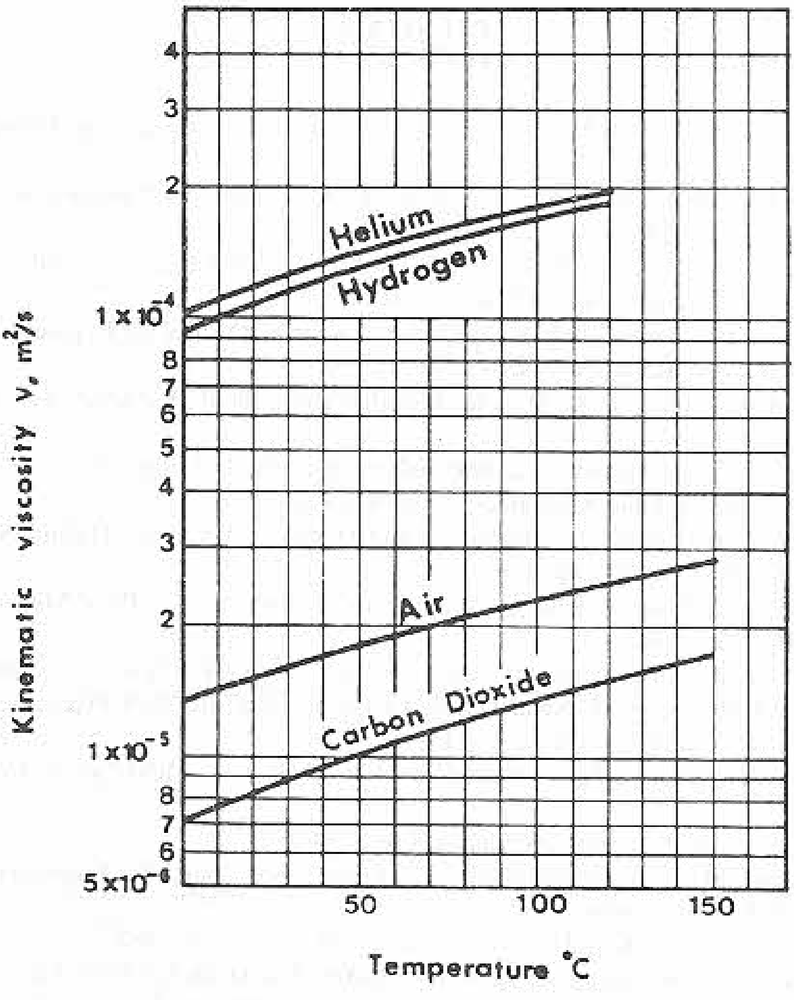


Fig. F.3 Kinematic viscosities of several gases

APPENDIX G
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APPENDIX H

NOMENCLATURE

A	area, m^2, also a constant (chapter 4)
a	acceleration, m/s^2, also speed of sound, m/s
$\vec{C}$	velocity vector, m/s
C	magnitude of the velocity vector, m/s
$\overline{C}$	average velocity, m/s
C_D	drag coefficient
C_L	lift coefficient
C_F	skin friction coefficient
c_p	specific heat at constant pressure
d	diameter, m
F	force, N
F_D	drag force, N
F_L	lift force, N
G	momentum, kg m/s
$\dot{G}$	rate of momentum, N
g	gravitational acceleration, m/s^2
H	constant height, m
$\overline{H}$	distance of the centre of pressure from one of the coordinates, m
h	variable height, m
K	fluid bulk modulus, Pa
k, k_i	constants (i = 1,2, ... etc.)
L_e	entrance length, m
M	moment, N.m
$M_{y,x}$	moment of force in direction x about y-axis
m	mass, kg
$\dot{m}$	mass flow rate, kg/s
$\vec{n}$	unit vector
P or p	static pressure, N/m^2
P_o	stagnation pressure, N/m^2
R	Gas constant, J/kg.K
Re	Reynolds number
Re_L	Reynolds number based on the length L
s	specific gravity

T	temperature, K
t	time, s
U	velocity of the wall (chapter 4 only), m/s
U_∞	velocity of free stream in direction x,m/s
u	velocity component in the x-direction, m/s
V	volume, m^3
$\dot{V}$	volume flow rate, m^3/s
v	velocity component in the y-direction, m/s,also specific volume, m^3/kg
w	velocity component in the z-direction, m/s
W	weight, Work
x	distance along the x-coordinate, m
y	distance along the y-coordinate, m
z	distance along the z-coordinate, m

Greek Letters

β	sum of external forces per unit mass, N/kg,also coefficient of compressibility, m^2/N
γ	specific weight, N/m^3 also, ratio of specific heat at constant pressure to that at constant volume for a gas
δ	thickness, also boundary layer thickness, m
δ_d	displacement thickness of boundary layer, m
θ	angle, degree
ν	kinematic viscosity, m^2/s
μ	dynamic viscosity, Ns/m^2
ρ	density, kg/m^3
τ	shear stress, N/m^2
σ	coefficient of surface tension
ω	angular velocity of a fluid particle, rad/s
Ω	angular speed, rad/s

Subscripts

f	fluid
fr	free surface
i	input
isent	isentropic
isoth	isothermal
l	left
m	manometer
r	right
o	output, also stagnation
st	surface tension

w	at wall
x	in direction x
y	in direction y
z	in direction z
1	location (1) in the fluid
2	location (2) in the fluid

Printed at King Abdulaziz University Press